A lower bound for the Lindelöf function associated to generalised integers[1]

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Abstract

In this paper we study generalised prime systems for which the integer counting function $N_{\mathcal{P}}(x)$ is asymptotically well-behaved, in the sense that $N_{\mathcal{P}}(x) = \rho x + O(x^\beta)$, where ρ is a positive constant and $\beta < \frac{1}{2}$. For such systems, the associated zeta function $\zeta_{\mathcal{P}}(s)$ has finite order for $\sigma = \Re s > \beta$, and the Lindelöf function $\mu_{\mathcal{P}}(\sigma)$ may be defined.

We prove that for all such systems, $\mu_{\mathcal{P}}(\sigma) \geq \mu_0(\sigma)$ for $\sigma > \beta$, where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ 0 & \text{if } \sigma \geq \frac{1}{2} \end{cases}.$$

Introduction

A *generalised prime system* (or *g-prime system*) $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying

$$1 < p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$$

and for which $p_n \to \infty$ as $n \to \infty$. From these can be formed the system $\mathcal{N}$ of *generalised integers* or *Beurling integers*; that is, the numbers of the form

$$p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$$

where $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{N}_0$.[2]

Such systems were first introduced by Beurling [3] and have been studied by many authors since then (see in particular [2]).

Much of the theory concerns connecting the asymptotic behaviour of the g-prime and g-integer counting functions, $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$, defined respectively by[3]

$$\pi_{\mathcal{P}}(x) = \sum_{p \in \mathcal{P}, p \leq x} 1 \quad \text{and} \quad N_{\mathcal{P}}(x) = \sum_{n \in \mathcal{N}, n \leq x} 1.$$

The methods invariably involve the associated *Beurling zeta function*, defined formally by

$$\zeta_{\mathcal{P}}(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} = \sum_{n \in \mathcal{N}} \frac{1}{n^s}. \tag{1}$$

In this paper, we shall be concerned with g-prime systems $\mathcal{P}$ for which

$$N_{\mathcal{P}}(x) = \rho x + O(x^\beta), \tag{2}$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. (For example, for the rational primes when $\mathcal{N} = \mathbb{N}$, this is true with $\beta = 0$ and $\rho = 1$.)

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[2] Here and henceforth, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P} = \{2, 3, 5, \ldots\}$ — the set of primes.

[3] We write $\sum_{p \in \mathcal{P}}$ to mean a sum over all the g-primes, counting multiplicities. Similarly for $\sum_{n \in \mathcal{N}}$.

For such systems, the product and series (1) converge for $\Re s > 1$ and $\zeta_{\mathcal{P}}(s)$ has an analytic continuation to the half-plane $\Re s > \beta$ except for a simple pole at $s = 1$ with residue ρ. Indeed, writing $N_{\mathcal{P}}(x) = \rho x + E(x)$ with $E(x) = O(x^{\beta})$, we have for $\Re s > 1$,

$$\begin{aligned}\zeta_{\mathcal{P}}(s) &= \int_{1-}^{\infty} x^{-s}\, dN_{\mathcal{P}}(x) = s\int_{1}^{\infty} \frac{N_{\mathcal{P}}(x)}{x^{s+1}}\, dx = s\int_{1}^{\infty} \frac{\rho x + E(x)}{x^{s+1}}\, dx \\ &= \frac{\rho s}{s-1} + s\int_{1}^{\infty} \frac{E(x)}{x^{s+1}}\, dx.\end{aligned}$$

The integral on the right converges for $\Re s > \beta$ and is an analytic function for such s.

Furthermore, $\zeta_{\mathcal{P}}(s)$ has *finite order* for $\Re s > \beta$; i.e. $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^{A})$ as $|t| \to \infty$ for some constant A for $\sigma > \beta$ (indeed, in our case this is true with $A = 1$). We can therefore define, as is usual, the *Lindelöf function* $\mu_{\mathcal{P}}(\sigma)$ to be the infimum of all real numbers λ such that $\zeta_{\mathcal{P}}(\sigma + it) = O(|t|^{\lambda})$. It is well-known that, as a function of σ, $\mu_{\mathcal{P}}(\sigma)$ is non-negative, decreasing, and convex (and hence continuous) (see, for example, [5]). Since $\mu_{\mathcal{P}}(\sigma) = 0$ for $\sigma > 1$, and (from above) $\mu_{\mathcal{P}}(\sigma) \le 1$ for $\sigma > \beta$, it follows by convexity that

$$\mu_{\mathcal{P}}(\sigma) \le \frac{1-\sigma}{1-\beta} \qquad \text{for } \beta < \sigma \le 1.$$

For $\mathcal{P} = \mathbb{P}$ (so that $\mathcal{N} = \mathbb{N}$), the Lindelöf Hypothesis is the conjecture that $\mu_{\mathbb{P}}(\sigma) = \mu_0(\sigma)$ for all σ, where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ 0 & \text{if } \sigma \ge \frac{1}{2} \end{cases}.$$

In this paper we prove that for all g-prime systems satisfying (2), $\mu_{\mathcal{P}}(\sigma)$ must be *at least* as large as $\mu_0(\sigma)$; i.e.

$$\mu_{\mathcal{P}}(\sigma) \ge \mu_0(\sigma) \qquad \text{for } \sigma > \beta.$$

This is, of course, trivial for $\sigma \ge \frac{1}{2}$, so we shall only concern ourselves with $\beta < \sigma < \frac{1}{2}$.

For the proof we employ the same methods (but strengthened) as those used in [4], where (essentially) it was shown that $\mu_{\mathcal{P}}(\sigma) > 0$ for any $\sigma < \frac{1}{2}$, in order to prove that for such systems we have $\psi_{\mathcal{P}}(x) - x = \Omega(x^{\frac{1}{2}-\delta})$ for every $\delta > 0$[4].

Main result

Theorem 1
Let $\mathcal{P}$ be a g-prime system for which

$$N_{\mathcal{P}}(x) = \rho x + O(x^{\beta}),$$

for some $\beta < \frac{1}{2}$ and $\rho > 0$. Let $\mu_{\mathcal{P}}(\sigma)$ and $\mu_0(\sigma)$ be as defined above. Then for $\sigma > \beta$, we have

$$\mu_{\mathcal{P}}(\sigma) \ge \mu_0(\sigma).$$

Proof. As mentioned above, we need only consider $\beta < \sigma < \frac{1}{2}$.

Suppose, for a contradiction, that we have $\mu_{\mathcal{P}}(\sigma) < \frac{1}{2} - \sigma$ for some $\sigma \in (\beta, \frac{1}{2})$. Then we can write

$$\mu_{\mathcal{P}}(\sigma) = \frac{1}{2} - \sigma - \delta,$$

[4]Here $\psi_{\mathcal{P}}(x)$ is the generalised Chebychev function: $\psi_{\mathcal{P}}(x) = \sum_{p^k \le x, p \in \mathcal{P}, k \in \mathbb{N}} \log p$ (counting multiplicities).

for some $\delta > 0$.

Let $\zeta_N(s) = \sum_{n \le N} n^{-s}$, where the sum ranges over $n \in \mathcal{N}$ (for clarity, we shall drop the subscript $\mathcal{P}$ throughout this proof). By identical arguments as those used in [4], we find that there exists constants $c_1, c_2 > 0$ such that for $R \ge c_1 N$,

$$\sum_{r=1}^{R} \int_0^{2r-1} |\zeta_N(\sigma + it)|^2 \, dt \ge c_2 R^2 N^{1-2\sigma}. \tag{3}$$

Also, writing $s = \sigma + it$, and following the arguments in [4], we have

$$\zeta_N(s) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta_{\mathcal{P}}(s+w) N^w}{w} \, dw + O\Big(\frac{N^c}{T(c+\sigma-1)}\Big) + O\Big(\frac{N^{1-\sigma}}{T} \sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|}\Big),$$

for $|t| < T$, $c > 1 - \sigma$ and $N \notin \mathcal{N}$.

Now push the contour in the integral to the left as far as $\Re w = -\eta$, where $\eta > 0$, picking up the residues at $w = 0$ and $w = 1 - s$ (since $|t| < T$). Here, η is chosen sufficiently small such that $\sigma - \eta > \beta$ and $\mu_{\mathcal{P}}(\sigma - \eta) < \frac{1}{2} - \sigma$. This is possible since $\mu_{\mathcal{P}}(\cdot)$ is continuous. Thus $\zeta_{\mathcal{P}}(\sigma - \eta + it) = O(|t|^{\frac{1}{2}-\sigma-\delta'})$ for some $\delta' > 0$.

The contribution along the horizontal line $[-\eta + iT, c + iT]$ is, in modulus, less than

$$\frac{1}{2\pi} \int_{-\eta}^{c} \frac{N^y |\zeta_{\mathcal{P}}(\sigma + y + i(t+T))|}{\sqrt{y^2 + T^2}} \, dy = O(N^c T^{-\frac{1}{2}-\sigma-\delta'}).$$

Similarly on $[-\eta - iT, c - iT]$. For the integral along $\Re w = -\eta$, we have

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{-\eta-iT}^{-\eta+iT} \frac{\zeta_{\mathcal{P}}(s+w) N^w}{w} \, dw \right| &\le \frac{N^{-\eta}}{2\pi} \int_{-T}^{T} \frac{|\zeta_{\mathcal{P}}(\sigma - \eta + i(t+y))|}{\sqrt{\eta^2 + y^2}} \, dy \\ &= O\Big(N^{-\eta} \int_{-T}^{T} \frac{T^{\frac{1}{2}-\sigma-\delta'}}{\sqrt{\eta^2 + y^2}} \, dy\Big) \\ &= O(N^{-\eta} T^{\frac{1}{2}-\sigma-\delta'} \log T). \end{aligned}$$

The residues at $w = 0$ and $w = 1 - s$ are, respectively, $\zeta_{\mathcal{P}}(s)$ and $\rho N^{1-s}/(1-s) = O(\frac{N^{1-\sigma}}{|t|+1})$. Putting these observations together and letting $c = 1 - \sigma + \frac{1}{\log N}$ (so that $N^c = eN^{1-\sigma}$), we have

$$\begin{aligned} \zeta_N(\sigma + it) = \zeta_{\mathcal{P}}(\sigma + it) &+ O\Big(\frac{N^{1-\sigma}}{|t|+1}\Big) + O(N^{1-\sigma} T^{-\frac{1}{2}-\sigma-\delta'}) + O(N^{-\eta} T^{\frac{1}{2}-\sigma-\delta'} \log T) \\ &+ O\Big(\frac{N^{1-\sigma} \log N}{T}\Big) + O\Big(\frac{N^{1-\sigma}}{T} \sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|}\Big), \end{aligned} \tag{4}$$

for $|t| < T$ and $N \notin \mathcal{N}$.

Fix $\alpha \in (0, \frac{1}{4\rho})$, and let $N \to \infty$ in such a way that $(N - \alpha, N + \alpha) \cap \mathcal{N} = \emptyset$. This is possible for if not, then $n' < n + 4\alpha$ (where n and n' are consecutive g-integers), which leads to $N(x) \gtrsim \frac{1}{4\alpha} x$ — a contradiction as $\frac{1}{4\alpha} > \rho$.

For such N, we can bound the final sum in (4) as follows. We have

$$\sum_{\substack{\frac{N}{2} < n < 2N \\ n \in \mathcal{N}}} \frac{1}{|n-N|} = \sum_{\substack{\alpha \le |n-N| < \sqrt{N} \\ n \in \mathcal{N}}} \frac{1}{|n-N|} + \sum_{\substack{\sqrt{N} \le |n-N| < \frac{N}{2} \\ n \in \mathcal{N}}} \frac{1}{|n-N|} + O(1)$$
$$= O\Big(N(N+\sqrt{N}) - N(N-\sqrt{N})\Big) + O\Big(\frac{N(\frac{3}{2}N)}{\sqrt{N}}\Big) + O(1)$$
$$= O(\sqrt{N}),$$

using $N(x) = \rho x + O(x^{\beta+\varepsilon})$ with $\beta < \frac{1}{2}$. (In fact, the better estimate $O(N^{\beta+\varepsilon})$ is possible by splitting the sum over smaller ranges, but $O(\sqrt{N})$ suffices for our purposes.) Hence (4) becomes

$$\zeta_N(\sigma+it) = \zeta_{\mathcal{P}}(\sigma+it) + O\Big(\frac{N^{1-\sigma}}{|t|+1}\Big) + O\Big(\frac{N^{1-\sigma}}{T^{\frac{1}{2}+\sigma+\delta'}}\Big) + O(N^{-\eta}T^{\frac{1}{2}-\sigma-\delta'}\log T) + O\Big(\frac{N^{\frac{3}{2}-\sigma}}{T}\Big). \quad (5)$$

Choosing $T = N^{1+\eta}$ makes the last three O-terms all $O(N^{\frac{1}{2}-\sigma-\eta'})$ for some $\eta' > 0$. Using the hypothetical bound $\zeta_{\mathcal{P}}(\sigma+it) = O(|t|^{\frac{1}{2}-\sigma-\delta'})$, (5) becomes

$$\zeta_N(\sigma+it) = O(|t|^{\frac{1}{2}-\sigma-\delta'}) + O\Big(\frac{N^{1-\sigma}}{|t|+1}\Big) + O(N^{\frac{1}{2}-\sigma-\eta'}).$$

Using the Cauchy-Schwarz inequality, we have

$$\sum_{r=1}^{R} \int_0^{2r-1} |\zeta_N(\sigma+it)|^2\, dt = O\Big(\sum_{r=1}^{R} \int_0^{2r-1} t^{1-2\sigma-2\delta'}\, dt\Big) + O\Big(\sum_{r=1}^{R} \int_0^{2r-1} \frac{N^{2-2\sigma}}{(t+1)^2}\, dt\Big)$$
$$+ O\Big(\sum_{r=1}^{R} \int_0^{2r-1} N^{1-2\sigma-2\eta'}\, dt\Big)$$
$$= O(R^{3-2\sigma-2\delta'}) + O(RN^{2-2\sigma}) + O(R^2N^{1-2\sigma-2\eta'}).$$

Taking R to be of slightly larger order than N, say $R = N\log N$, the RHS becomes $o(R^2N^{1-2\sigma})$, which contradicts (3).

□

Remark. The result is best possible — at least if we assume the Lindelöf Hypothesis. If $\mathcal{P} = \mathbb{P}$, then (2) holds with $\beta = 0$ and, on the Lindelöf Hypothesis, $\mu_{\mathcal{P}} = \mu_0$. However, it is conceivable that the result might be subject to further improvements if (2) holds with $\beta > 0$. The example below shows this is not the case — again on the assumption of the Lindelöf Hypothesis.

Let $\beta \in (0, \frac{1}{2})$ and denote by $\mathcal{P}$ the g-prime system made up of p and $p^{1/\beta}$ where p varies over all the primes; i.e.

$$\mathcal{P} = \mathbb{P} \cup \{p^{\frac{1}{\beta}} : p \in \mathbb{P}\}.$$

For this system, $N_{\mathcal{P}}(x)$ satisfies (2). Indeed,

$$N_{\mathcal{P}}(x) = \sum_{n \le x^\beta} \Big[\frac{x}{n^{1/\beta}}\Big] = \sum_{n \le a^\beta} \Big[\frac{x}{n^{1/\beta}}\Big] + \sum_{n \le b} \Big[\Big(\frac{x}{n}\Big)^\beta\Big] - [a^\beta][b],$$

for any $ab = x$ (see [1] for such manipulations). Putting $a = x^\lambda$, we obtain

$$\begin{aligned}
N_{\mathcal{P}}(x) &= x \sum_{n \le x^{\lambda\beta}} \frac{1}{n^{1/\beta}} + x^\beta \sum_{n \le x^{1-\lambda}} \frac{1}{n^\beta} - x^{\lambda\beta+1-\lambda} + O(x^{\lambda\beta}) + O(x^{1-\lambda}) \\
&= x\left(\zeta\Big(\frac{1}{\beta}\Big) - \frac{\beta}{1-\beta} x^{-\lambda\beta(\frac{1}{\beta}-1)} + O(x^{-\lambda\beta(\frac{1}{\beta})})\right) \\
&+ x^\beta\Big(\frac{x^{(1-\lambda)(1-\beta)}}{1-\beta} + \zeta(\beta) + O(x^{-(1-\lambda)\beta}\Big) - x^{\lambda\beta+1-\lambda} + O(x^{\lambda\beta}) + O(x^{1-\lambda}) \\
&= \zeta\Big(\frac{1}{\beta}\Big)x + \zeta(\beta)x^\beta + O(x^{\lambda\beta}) + O(x^{1-\lambda}).
\end{aligned}$$

Choosing $\lambda = \frac{1}{1+\beta}$ so that $\lambda\beta = 1 - \lambda$ minimises the error. This gives

$$N_{\mathcal{P}}(x) = \zeta\Big(\frac{1}{\beta}\Big)x + \zeta(\beta)x^\beta + O(x^{\frac{\beta}{1+\beta}}).$$

The associated Beurling zeta function is $\zeta(s)\zeta(s/\beta)$. On the Lindelöf Hypothesis, it follows that $\mu_{\zeta(\cdot/\beta)}(\sigma) = 0$ for $\sigma \ge \frac{\beta}{2}$. Thus $\mu_{\mathcal{P}}(\sigma) \le \frac{1}{2} - \sigma$ for $\beta < \sigma < \frac{1}{2}$. By Theorem 1, we must have $\ge$ as well, so in fact there is equality; i.e.

$$\mu_{\mathcal{P}}(\sigma) = \frac{1}{2} - \sigma,$$

for $\beta < \sigma < \frac{1}{2}$.

References

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