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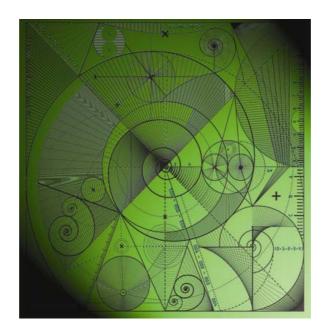
Is the Helmholtz equation really sign-indefinite?

by

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Abstract

The usual variational (or weak) formulations of the Helmholtz equation are sign-indefinite in the sense that the bilinear forms cannot be bounded below by a positive multiple of the appropriate norm squared. This is often for a good reason, since in bounded domains under certain boundary conditions the solution of the Helmholtz equation is not unique at certain wavenumbers (those that correspond to eigenvalues of the Laplacian), and thus the variational problem cannot be sign-definite. However, even in cases where the solution is unique for all wavenumbers, the standard variational formulations of the Helmholtz equation are still indefinite when the wavenumber is large. This indefiniteness has implications for both the analysis and the practical implementation of finite element methods. In this paper we introduce new *sign-definite* (also called *coercive* or *elliptic*) formulations of the Helmholtz equation posed in either the interior of a star-shaped domain with impedance boundary conditions, or the exterior of a star-shaped domain with Dirichlet boundary conditions. Like the standard variational formulations, these new formulations arise just by multiplying the Helmholtz equation by particular test functions and integrating by parts.

1   Introduction

The Helmholtz equation,

$$\Delta u + k^2 u = 0, \tag{1.1}$$

with wavenumber $k > 0$, is arguably the simplest possible model of wave propagation. For example, if we look for solutions of the wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0 \tag{1.2}$$

in the form $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$, then the function $u(\mathbf{x})$ satisfies the Helmholtz equation (1.1) with $k = \omega/c$. Assuming a similar dependence on time reduces the Maxwell equations to the so-called time-harmonic Maxwell equations, and in certain situations these can be further reduced to the Helmholtz equation. Similarly, the time-harmonic elastic wave equation (often called the Navier equation) also reduces to the Helmholtz equation in certain circumstances. Because the Helmholtz equation is at the heart of linear wave propagation, much research effort has gone into both studying the properties of its solutions (for example their asymptotic behaviour as $k \to \infty$) and designing methods for computing them efficiently.

Many numerical methods for solving the Helmholtz equation are based on its standard variational (or weak) formulations, and these are sign-indefinite when k is large. In the literature, one often finds this sign-indefiniteness attributed to the Helmholtz equation itself; some recent examples of this attribution include the following

> "...the Helmholtz operator for scattering problems is a highly indefinite complex-valued linear operator." (2012)

> "The main difficulty of the analysis is caused by the strong indefiniteness of the Helmholtz equation." (2009)

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"Problems in high-frequency scattering of acoustic or electromagnetic waves are highly indefinite." (2012)

The goal of this paper is to introduce new *sign-definite* variational formulations of two frequently-encountered boundary value problems (BVPs) for the Helmholtz equation. These formulations can be obtained by multiplying by particular test functions and integrating by parts (just like the standard formulations). Thus, we aim to emphasise that, whereas the standard variational formulations of the Helmholtz equation are sign-indefinite, this sign-indefiniteness is *not* an inherent feature of the Helmholtz equation, only of its standard formulations.

1.1 Background: variational formulations of the Helmholtz equation

One of the most common variational problems is the following: given a Hilbert space $\mathcal{V}$, a bilinear form $a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and a continuous linear functional $F : \mathcal{V} \to \mathbb{R}$,

$$\text{find } u \in \mathcal{V} \text{ such that } a(u,v) = F(v) \text{ for all } v \in \mathcal{V}. \tag{1.3}$$

The particular variational problem that most mathematicians first encounter is that corresponding to the Dirichlet problem for Poisson's equation, i.e. given a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and a real function f on Ω, find u such that

$$\Delta u = -f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

The associated variational problem is given by (1.3) with

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} \quad \text{and} \quad F(v) := \int_\Omega f \, v \, d\mathbf{x}, \tag{1.5}$$

where the Hilbert space is $H_0^1(\Omega)$ (informally, functions in the Sobolev space $H^1(\Omega)$ that are zero on $\partial\Omega$) with inner product and norm

$$(u,v)_{H_0^1(\Omega)} := \int_\Omega (\nabla u \cdot \nabla v + uv) \, d\mathbf{x}, \quad \|v\|_{H_0^1(\Omega)}^2 := \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \,.$$

This variational formulation is obtained by multiplying the PDE in (1.4) by a $v \in H_0^1(\Omega)$, integrating over Ω, and using Green's first identity

$$\int_\Omega v \Delta u \, d\mathbf{x} = -\int_\Omega \nabla v \cdot \nabla u \, d\mathbf{x} + \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds, \tag{1.6}$$

i.e. the divergence theorem applied to $v \nabla u$.

Returning to the general variational problem (1.3), ideally one would like to prove that there exist $C_c, \alpha > 0$ such that

$$|a(u,v)| \le C_c \|u\|_\mathcal{V} \|v\|_\mathcal{V} \quad \text{for all } u, v \in \mathcal{V}, \qquad \text{(continuity)}, \tag{1.7}$$

$$|a(v,v)| \ge \alpha \|u\|_\mathcal{V}^2 \quad \text{for all } v \in \mathcal{V}, \qquad \text{(coercivity)}. \tag{1.8}$$

"Sign-definite" is used as a synonym for "coercive" (thus a variational problem is sign-indefinite if and only if it is not coercive). Note that several authors call property (1.8) "$\mathcal{V}$-ellipticity" (see, e.g., [40, §2.4.1], [16, §1], [61, Equation 2.43]) and use the word "coercivity" for the weaker property of satisfying a Gårding inequality ([40, §2.4.3], [61, Definition 2.1.54]). If the two properties (1.7) and (1.8) can be established then there are three important consequences. The first is that the Lax–Milgram theorem implies that there exists a unique solution to (1.3), and this satisfies

$$\|u\|_\mathcal{V} \le \frac{1}{\alpha} \|F\|_{\mathcal{V}'} \,. \tag{1.9}$$

The second and third consequences concern the Galerkin discretisation of the variational problem (1.3), namely, given $\mathcal{V}_N$ a finite dimensional subspace of $\mathcal{V}$ (with dimension N),

$$\text{find } u_N \in \mathcal{V}_N \text{ such that } a(u_N, v_N) = F(v_N) \text{ for all } v_N \in \mathcal{V}_N. \tag{1.10}$$

If continuity (1.7) and coercivity (1.8) hold then the Lax–Milgram theorem implies that the Galerkin solution u_N exists and is unique, and Céa's Lemma implies that u_N satisfies

$$\|u - u_N\|_{\mathcal{V}} \leq \frac{C_c}{\alpha} \inf_{w_N \in \mathcal{V}_N} \|u - w_N\|_{\mathcal{V}}, \tag{1.11}$$

where C_c and α are as in (1.7) and (1.8) respectively (see, e.g., [8, §2.8]); u_N is then said to be *quasi-optimal*. The third consequence is that the finite dimensional matrix of the Galerkin method, $\mathbf{A}$, inherits analogous continuity and coercivity properties from the bilinear form:

$$|(\mathbf{A}\mathbf{u}, \mathbf{v})| \leq M_2 C_c \|\mathbf{u}\| \|\mathbf{v}\| \quad \text{and} \quad |(\mathbf{A}\mathbf{v}, \mathbf{v})| \geq M_1 \alpha \|\mathbf{v}\|^2, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^N, \tag{1.12}$$

where M_1 and M_2 are constants depending on the discretisation (see Section 5.2). Coercivity in particular has important implications for the efficient solution of the linear system involving this matrix.

For the Dirichlet problem for Poisson's equation (1.4), continuity of $a(\cdot, \cdot)$ follows from the Cauchy–Schwarz inequality, and coercivity follows from the Poincaré–Friedrichs inequality (namely $\|v\|_{H^1(\Omega)} \leq c \|\nabla v\|_{L^2(\Omega)}$ for some $c > 0$ for all $v \in H_0^1(\Omega)$, see, e.g., [8, §5.3], [29, §5.6.1]). Thus the variational problem (1.5) has a unique solution and the Galerkin equations (1.10) have a unique, quasi-optimal solution for any subspace $\mathcal{V}_N \subset H_0^1(\Omega)$. Furthermore, the fact that $a(\cdot, \cdot)$ is coercive and also symmetric (i.e. $a(u, v) = a(v, u)$) means that the linear system arising from the Galerkin method is positive definite, and thus can be solved efficiently by iterative solvers such as the conjugate gradient method, or multigrid (see, e.g., the textbooks [24, Chapter 2], [7]).

In contrast, the situation for the Dirichlet problem for the Helmholtz equation:

$$\Delta u + k^2 u = -f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \tag{1.13}$$

with $k > 0$ and f a given real function, is very different. Indeed, the BVP (1.13) does not have a unique solution if $k^2 = \lambda_j$ for λ_j an eigenvalue of the negative Laplacian in Ω with zero Dirichlet boundary conditions. Proceeding as before, we multiply the PDE in (1.13) by $v \in H_0^1(\Omega)$, integrate over Ω and use Green's first identity, and obtain the variational problem (1.3) with $F(\cdot)$ as in (1.5) but $a(\cdot, \cdot)$ now given by

$$a(u, v) := \int_{\Omega} \left(\nabla u \cdot \nabla v - k^2 u v \right) \, \mathrm{d}\mathbf{x}. \tag{1.14}$$

For the Helmholtz equation it is convenient to use the k-dependent inner product and norm

$$(u, v)_{1,k,\Omega} := \int_{\Omega} \left(\nabla u \cdot \nabla v + k^2 u v \right) \, \mathrm{d}\mathbf{x}, \quad \|v\|_{1,k,\Omega}^2 := \|\nabla v\|_{L^2(\Omega)}^2 + k^2 \|v\|_{L^2(\Omega)}^2 \tag{1.15}$$

on the space $H_0^1(\Omega)$. Continuity of $a(\cdot, \cdot)$ follows as before using the Cauchy–Schwarz inequality, but now

$$a(v, v) = \int_{\Omega} |\nabla v|^2 \, \mathrm{d}\mathbf{x} - k^2 \int_{\Omega} |v|^2 \, \mathrm{d}\mathbf{x}.$$

It is clear that $a(v, v)$ cannot be bounded below by $\|v\|_{1,k,\Omega}^2$; indeed, if $k^2 = \lambda_j$ (the j-th eigenvalue of the negative Laplacian with Dirichlet boundary conditions) then $a(u_j, u_j) = 0$, for u_j the corresponding eigenfunction. One can also show that if $k^2 > \lambda_1$ then the bilinear form takes both positive and negative real values, namely $a(u_1, u_1) < 0 < a(u_j, u_j)$ where j is chosen such that $k^2 < \lambda_j$; thus the bilinear form is not coercive by [4, Propositions 3.2 and 3.3].

Although $a(\cdot, \cdot)$ is not coercive, it satisfies a Gårding inequality, i.e. adding a multiple of $\|v\|_{L^2(\Omega)}^2$ to $a(v, v)$ makes it coercive, since

$$a(v, v) + 2k^2 \|v\|_{L^2(\Omega)}^2 = \|v\|_{1,k,\Omega}^2. \tag{1.16}$$

Even though we no longer have coercivity, can we recover any of its three consequences described above (existence and uniqueness, quasi-optimality, and sign-definiteness of the discretised linear system)? Classic Fredholm theory implies that if k^2 is not an eigenvalue of the negative Laplacian, then a solution to the variational problem (1.3) exists and is unique (this relies on the fact that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, see, e.g., [29, §6.2.3], [61, Theorem 2.10.4]), but, although

this method does give a bound on u in terms of f, this bound is not explicit in k. One can also show that, given a suitable finite dimensional subspace $\mathcal{V}_N$, the Galerkin equation (1.10) has a solution which satisfies

$$\|u - u_N\|_{\mathcal{V}} \leq \widetilde{C} \inf_{w_N \in \mathcal{V}_N} \|u - w_N\|_{\mathcal{V}} \tag{1.17}$$

for some $\widetilde{C} > 0$, *provided the subspace dimension N is large enough* (see, e.g., [61, Theorem 4.2.9]). However, it is very difficult to find out how the threshold for N and constant $\widetilde{C}$ in (1.17) depend on k. Finally, the Galerkin matrix for this problem is still symmetric, as in the Poisson case, but is no longer positive definite, having both positive and negative eigenvalues when k^2 is sufficiently large. This fact, coupled also with difficulties if k^2 is close to an eigenvalue of the Laplacian, mean that it is harder to solve the linear system arising from the Helmholtz bilinear form (1.14) than the Poisson one (1.5).

Although the Helmholtz equation in Ω with Dirichlet boundary conditions is not well-posed for every k, the solution under impedance boundary conditions, i.e.

$$\Delta u + k^2 u = -f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} - \mathrm{i}ku = g \quad \text{on } \partial\Omega, \tag{1.18}$$

where f and g are given functions, exists and is unique for every real $k \neq 0$. (This is because the eigenvalues of the Laplacian with impedance boundary conditions are not real.) How will considering the Helmholtz equation under impedance boundary conditions instead of Dirichlet boundary conditions change the properties of the associated variational formulation? One immediate difference is that, since the boundary conditions involve i, the variational formulation of this BVP involves a sesquilinear form $a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ instead of a bilinear form, and the functional $F(\cdot)$ can now take values in $\mathbb{C}$ (and must be antilinear). Multiplying the PDE in (1.18) by $\overline{v}$, integrating over Ω, and using Green's first identity and the impedance boundary condition, we obtain the variational problem (1.3) with

$$a(u, v) := \int_{\Omega} \left(\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v} \right) \mathrm{d}\mathbf{x} - \mathrm{i}k \int_{\partial\Omega} u\overline{v} \, \mathrm{d}s, \qquad F(v) := \int_{\Omega} f\overline{v} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} g\overline{v} \, \mathrm{d}s, \tag{1.19}$$

and the appropriate Hilbert space now $H^1(\Omega)$ with norm and inner product (1.15) (replacing v by $\overline{v}$ in the integral). Continuity of $a(\cdot, \cdot)$ follows in a similar way to before (although since $a(\cdot, \cdot)$ now involves an integral over $\partial\Omega$ we also need to use the fact that the trace map is continuous). The arguments that show that $a(\cdot, \cdot)$ for the Dirichlet problem is not coercive also show that $a(\cdot, \cdot)$ in (1.19) is not coercive for $k^2 \geq \lambda_1$ (this is because the integral over $\partial\Omega$ in $a(v, v)$ is zero if v is a Dirichlet eigenfunction of the negative Laplacian). Since

$$\Re a(v, v) = \int_{\Omega} |\nabla v|^2 \, \mathrm{d}\mathbf{x} - k^2 \int_{\Omega} |v|^2 \, \mathrm{d}\mathbf{x},$$

the real part of $a(\cdot, \cdot)$ satisfies (1.16), and thus $a(\cdot, \cdot)$ satisfies a Gårding inequality. Fredholm theory can then be applied as in the case of the Dirichlet problem to show that a solution to the variational problem exists, and that given a finite dimensional subspace $\mathcal{V}_N$ the Galerkin solution u_N exits, is unique, and satisfies (1.17), provided that N is greater than some threshold. Again, this classic theory gives no information about the how the constants depend on k, but this dependence has been quantified using different techniques in [44, Proposition 8.2.7] (for the h-version of the finite element method) and [47], [46] (for the hp-version). Finally, regarding the linear system: this is sign-indefinite as in the Dirichlet case, and furthermore, because the boundary condition involves i, $a(u, v) \neq a(v, u)$; thus the eigenvalues are complex and lie on both sides of the imaginary axis. These facts are not the only reasons why it is difficult to solve the linear systems associated with Helmholtz problems, but they contribute strongly to this difficulty; see the reviews [26], [27], [25], [1] and the references therein for more details.

In summary, in moving from Dirichlet boundary conditions to impedance boundary conditions, even though we gain well-posedness of the Helmholtz equation for every k, *we still keep the sign-indefiniteness of the sesquilinear form*. The main aim of this paper is to show that it *is* possible to have a sign-definite, i.e. coercive, formulation of the Helmholtz equation under impedance boundary

conditions (at least for a wide class of domains, namely, star-shaped domains), if one is prepared to modify the space $\mathcal{V}$ and the sesquilinear form $a(\cdot, \cdot)$.

We note at this stage that other coercive formulations of the Helmholtz impedance problem do exist. We discuss these in more detail in §1.3 below, but emphasise here that for these formulations at least one of the following is true: *(i)* the formulation is an integral equation on $\partial\Omega$ *(ii)* the formulation requires restricting $\mathcal{V}$ to include only (piecewise) solutions of the homogeneous Helmholtz equation (so-called "operator-adapted" or "Trefftz" spaces), *(iii)* the formulation is a least-squares formulation (under which *any* well-posed PDE is coercive). In contrast, the formulation introduced in this paper is a formulation in Ω (not on the boundary $\partial\Omega$), does not require operator-adapted spaces, and is not a least-squares formulation.

1.2 A new coercive variational formulation of the Helmholtz equation

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$ consider the problem of solving the Helmholtz equation in Ω subject to an impedance boundary condition:

$$\Delta u + k^2 u = -f \quad \text{in } \Omega, \tag{1.20a}$$

$$\frac{\partial u}{\partial n} - \mathrm{i}ku = g \quad \text{on } \partial\Omega. \tag{1.20b}$$

In this paper we also consider the sound-soft scattering problem for the Helmholtz equation, i.e. (1.20a) posed in the *exterior* of a bounded domain with Dirichlet boundary conditions, and the results outlined below for the interior impedance problem have counterparts for the exterior problem. For simplicity, however, we just discuss the interior problem in this introduction (the exterior problem is discussed in §4).

Consider the Hilbert space

$$V := \left\{ v : v \in H^1(\Omega), \ \Delta v \in L^2(\Omega), \ \nabla v \in \left(L^2(\partial\Omega) \right)^d \right\} \tag{1.21}$$

with norm

$$\|v\|_V^2 := k^2 \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + k^{-2} \|\Delta v\|_{L^2(\Omega)}^2 + L \left(k^2 \|v\|_{L^2(\partial\Omega)}^2 + \|\nabla v\|_{L^2(\partial\Omega)}^2 \right), \tag{1.22}$$

and obvious inner product, where L is the diameter (i.e. a characteristic length scale) of the domain. (We weight the derivatives by k and include L in front of the boundary terms so that, when computed for solutions of the homogeneous Helmholtz equation with wavenumber k, each term of the norm scales in the same way as k and L vary—see Remark 3.8 below.)

Although this space may appear strange, standard regularity results imply that if $u \in H^1(\Omega)$ is the solution to (1.20) then $u \in V$; see Proposition 3.2 below. In addition, we show below that $V \subset H^{3/2}(\Omega)$, and for classical solutions of the homogeneous Helmholtz equation these two spaces are in fact equivalent (i.e. if $v \in C^2(\Omega)$ is such that $\Delta v + k^2 v = 0$, then $v \in H^{3/2}(\Omega)$ implies that $v \in V$); see Remark 3.7.

Define the sesquilinear form $b : V \times V \to \mathbb{C}$ by

$$b(u, v) := \int_\Omega \left(\nabla u \cdot \overline{\nabla v} + k^2 u \overline{v} + \left(\mathcal{M}u + \frac{1}{3k^2} \mathcal{L}u \right) \overline{\mathcal{L}v} \right) \mathrm{d}\mathbf{x} \tag{1.23}$$

$$- \int_{\partial\Omega} \left(\mathrm{i}ku \overline{\mathcal{M}v} + \left(\mathbf{x} \cdot \nabla_{\partial\Omega} u - \mathrm{i}k\beta u + \frac{d-1}{2} u \right) \frac{\overline{\partial v}}{\partial n} + (\mathbf{x} \cdot \mathbf{n}) \left(k^2 u \overline{v} - \nabla_{\partial\Omega} u \cdot \overline{\nabla_{\partial\Omega} v} \right) \right) \mathrm{d}s,$$

and functional $G : V \to \mathbb{C}$ by

$$G(v) := \int_\Omega \left(\overline{\mathcal{M}v} - \frac{1}{3k^2} \overline{\mathcal{L}v} \right) f \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \overline{\mathcal{M}v} \, g \, \mathrm{d}s, \tag{1.24}$$

where β is an arbitrary real constant, d is the spatial dimension,

$$\mathcal{L}u := \Delta u + k^2 u, \quad \mathcal{M}u := \mathbf{x} \cdot \nabla u - \mathrm{i}k\beta u + \frac{d-1}{2} u,$$

and $\nabla_{\partial\Omega}$ is the surface gradient on $\partial\Omega$ (recall that $\nabla_{\partial\Omega}$ is such that if u is differentiable in a neighbourhood of $\partial\Omega$ then

$$\nabla_{\partial\Omega} u = \nabla u - \mathbf{n}\frac{\partial u}{\partial n}$$

on $\partial\Omega$).

The sesquilinear form $b(\cdot,\cdot)$ and functional $G(\cdot)$ are defined in this way because if u is the solution to the BVP (1.20), then

$$b(u,v) = G(v) \quad \text{for all } v \in V; \tag{1.25}$$

this is not obvious, and we explain why below (in §1.4 and Proposition 3.2).

Using the Cauchy–Schwarz inequality it is straightforward to show that the sesquilinear form $b(\cdot,\cdot)$ is continuous on V, i.e. (1.7) holds with $\mathcal{V} = V$. In particular, if β is independent of k (as we choose it to be below), $C_c \sim k$ as $k \to \infty$; the explicit value of the constant C_c is described in Lemma 3.3 below.

The main novelty of $b(\cdot,\cdot)$ is that, for some domains, it is coercive on V:

Theorem 1.1. *Let Ω be a Lipschitz domain with diameter L that is star-shaped with respect to a ball, i.e. there exists a $\gamma > 0$ such that*

$$\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq \gamma L$$

for all $\mathbf{x} \in \partial\Omega$ such that $\mathbf{n}(\mathbf{x})$ exists (see Remark 3.5 for how this is related to the usual definition of star-shapedness). If the arbitrary constant β is chosen such that

$$\beta \geq \frac{L}{2}\left(1 + \frac{4}{\gamma} + \frac{\gamma}{2}\right)$$

then, for any $k > 0$,

$$\Re b(v,v) \geq \frac{\gamma}{4}\|v\|_V^2 \quad \text{for all } v \in V,$$

i.e. $b(\cdot,\cdot)$ is coercive on V with constant $\gamma/4$.

Following the discussion in §1.1 we know there are three immediate consequences of this result:

1. The variational problem (1.25) has a unique solution which satisfies $\|u\|_V \leq (4/\gamma)\|G\|_{V'}$.

2. The Galerkin method applied to (1.25) has a unique, quasi-optimal solution for *any* finite dimensional subspace $V_N \subset V$, with an explicit bound for the constant of quasi-optimality given by $4C_c/\gamma$.

3. The linear system resulting from the Galerkin method is also coercive (in the sense of (1.12)) with an explicit value for the coercivity constant.

Regarding **1.**: this is the least interesting consequence, since we already have existence and uniqueness of the solution to the BVP (1.20) from the standard variational formulation and Fredholm theory (although it is perhaps interesting that we can get these results in this alternative way). It is straightforward to bound $\|G\|_{V'}$ in terms of the L^2-norms of f and g; see Remark 3.6. However, the resulting bound on $\|u\|_V$ was already essentially proved in [44, Proposition 8.1.4] for $d = 2$ and [18, Theorem 1] for $d = 3$.

Regarding **2.**: this is interesting because, as discussed in §1.1, establishing quasi-optimality of the Galerkin method for the standard variational formulation (1.19) with all the constants (including the threshold for quasi-optimality to hold) explicit in k is a challenging problem. Note that for the standard variational formulation (1.19) there are in fact two k-dependent thresholds for the subspace dimension N: one for the bound (1.17) to hold, and then one for the best approximation error on the right-hand side to be small (the latter depends on the particular V_N and is a consequence of the fact that solutions of the Helmholtz equation are highly oscillatory). The new formulation eliminates the first threshold, but the second one still remains (since it is a consequence of approximation theory and independent of the variational formulation).

The main disadvantage of the new formulation is that the space V includes the requirement $\Delta v \in L^2(\Omega)$. This means that the standard C^0 finite element spaces of $H^1(\Omega)$ are not subspaces of V, and in fact, any finite element space that is a subspace of V (i.e. the elements are conforming) must also be a subspace of $C^1(\overline{\Omega})$ (see §5.1). Of course, there are several well-known piecewise-polynomial finite element spaces consisting of C^1-elements (originally designed for solving the biharmonic equation) that could then be used in the new formulation to give a conforming method; we discuss this more in Section 5.

Regarding **3.**: as discussed briefly in §1.1, solving the Helmholtz equation with iterative methods is difficult, and a contributing factor is the sign-indefiniteness of the standard variational formulations. Whether the new formulation can alleviate some of this difficulty remains to be seen and will require a detailed, separate investigation. However, as a start, in §5.2 we investigate whether we can determine anything a priori about how the Generalised Minimal Residual method (GMRES) behaves when it is applied to the linear systems arising from new formulation (without any preconditioning).

1.3   Other coercive formulations

In §1.1 we discussed the most basic variational formulation of the interior impedance problem (1.19) and saw that this formulation was sign-indefinite. Of course, there are many different ways of formulating BVPs involving the Helmholtz equation; the vast majority of these, however, are also sign-indefinite. There do in fact exist a few coercive formulations, which we now briefly outline. As explained at the end of §1.1, these coercive formulations are very different from the ones introduced in this paper. We also discuss some formulations that are not coercive, but enjoy some of the benefits of coercivity.

Integral equation formulations   Since closed-form expressions for the fundamental solution of the Helmholtz equation exist, a popular way of solving BVPs such as (1.20) is by reformulating them as integral equations on the boundary $\partial\Omega$ (this approach is especially popular when considering problems posed in unbounded domains).

- The standard second-kind integral operator used to solve the Dirichlet problem in the exterior of a bounded obstacle (the so-called "combined potential" or "combined field" operator for this problem) is coercive for a variety of domains when k is large enough [21, Theorems 4.2 and 4.12], [4], [63], [12, §5.7]. By standard properties of integral equations, this integral operator can also be used to solve the interior impedance problem (1.20) (see, e.g., [12, Corollary 2.28 and Theorem 2.30])

- A modification of the standard combined potential operator for the exterior Dirichlet problem, the so-called "star-combined operator" is coercive for all $k > 0$ for all Lipschitz domains that are star-shaped with respect to a ball [62].

- A modification of the standard combined potential operator for the exterior Neumann problem is coercive for the circle and sphere when k is large enough [6].

- In the case of scattering from a flat screen, the standard first kind integral equations for both the Dirichlet and Neumann problems are coercive for all $k > 0$ [34], [13].

Trefftz-discontinuous Galerkin methods   Since approximating highly oscillatory solutions of the Helmholtz equation with piecewise polynomials requires large numbers of degrees of freedom, many methods have been proposed that seek to approximate solutions of the Helmholtz equation with oscillatory basis functions. One of the main classes of these "wave-based" methods are *Trefftz methods*, which use basis functions that are locally (i.e. inside each mesh element) solutions of the Helmholtz equation. One of the main examples of such a method is the Ultra Weak Variational Formulation (UWVF) [10], [11], which can be recast as a special discontinuous Galerkin (DG) method.

For such "Trefftz-discontinuous Galerkin methods" applied to either the interior impedance problem (1.20) or the exterior Dirichlet problem as formulated in Definition 4.2 below, the associated sesquilinear form is continuous and coercive in a norm consisting of jumps of functions over

element edges [38, §3.1], [49, §4.3], [9, Lemma 3.4] (a slightly weaker result was proved in the original analysis of the UWVF, see [11, Lemma 3.3, eq. (3.30)]). Error estimates in a mesh-independent norm (such as the $L^2(\Omega)$ norm) can then be obtained by using a duality argument.

Least-squares methods   As we saw in §1.1, the best possible variational problem involves a symmetric, coercive, sesquilinear form, as in the case of the Poisson equation. Least-squares finite elements can be viewed as an attempt to recover this situation for non-symmetric or indefinite problems; indeed, the standard least-squares formulation of *any* well-posed BVP for *any* linear PDE (with linear boundary conditions) leads to a symmetric, coercive, sesquilinear form [5, §2.2.1, §3.2]. This is not the end of the story, however, since there are then subtle questions about which norms to choose for the least-squares functionals.

In the least-squares framework, second order PDEs are usually converted into first-order systems to reduce the condition number of the discretised problem. A standard first-order system reformulation of the Helmholtz exterior Dirichlet problem was considered in [42]. The authors proved that this formulation was well-posed, and hence coercive, but did not determine how the coercivity constant depends on k; this dependence can in principle be determined using the k-explicit bounds on the solution of the Helmholtz equation that have recently been obtained.

A new variational formulation of the Helmholtz equation as a first-order system was recently introduced in [20]. This "Discontinuous Petrov Galerkin (DPG)" method can be thought of as a least-squares method in a non-standard inner product. Using k-explicit bounds on the solution of the Helmholtz interior impedance problem, a fully k-explicit analysis of the "theoretical" version of this method is given in [20], whereas a k-explicit analysis of the "practical" version is still lacking. For this latter version, the matrix of the Galerkin discretisation is only positive-semidefinite instead of positive-definite.

T-coercivity   Any well-posed variational problem of the form (1.3) is coercive *if* one is allowed to introduce another bounded linear operator into the sesquilinear form. That is, if the variational problem (1.3) has a unique solution that depends continuously on $F(\cdot)$ (or equivalently $a(\cdot, \cdot)$ satisfies an inf-sup condition [61, §2.1.6]), then there exists a $T : \mathcal{V} \to \mathcal{V}$ and an $\alpha' > 0$ such that

$$|a(v, Tv)| \geq \alpha' \|v\|_{\mathcal{V}}^2 \quad \text{for all } v \in \mathcal{V};$$

see [61, Remark 2.1.48, Page 39], [15, Theorem 1]. This reformulation only yields the advantages of coercivity, however, if the variational problem is sufficiently simple for T to be known explicitly. In the case of the standard variational formulation of the Helmholtz Dirichlet problem, the operator T can be expressed in terms of eigenspace projectors and thus approximated by discrete operators on sufficiently fine meshes; however, the size of the meshwidth threshold is not clear [17, §3].

Interior penalty methods   Finally, recall that interior penalty methods consist of adding terms to the appropriate sesquilinear forms to penalise jumps of various quantities over interfaces between elements of a mesh. Although the variational formulations of these methods are not coercive, for certain methods some of the consequences of coercivity hold; namely, the Galerkin equation has a unique solution without any constraint on the dimension of the (piecewise polynomial) approximation space, and error estimates can be obtained that are explicit in k, h, and p [31, Remarks 4.3 and 5.1], [32, Remark 3.2], [66, Corollary 3.5, Theorem 4.4]. For the interior penalty discontinuous Galerkin (IPDG) methods introduced in [31], [32] the penalty terms are added so that the properties just highlighted can be proved using Rellich-type identities. (These methods share a conceptual link with the new variational formulations introduced in this paper, since, as we see in §1.4 below, the new formulations in this paper are designed using closely-related Morawetz-type identities.) Adding a penalty term to the standard variational formulation (1.19) was considered in [66]. For this formulation the properties above are proved for subspaces consisting of piecewise-linear polynomials using the fact that functions in these subspaces satisfy Laplace's equation on each element, and then using Green's identity for Laplace's equation (i.e. (1.6)).

1.4 The idea behind the new formulation

As we saw in §1.1, the standard variational formulation of the interior impedance BVP for the Helmholtz equation (1.20) is based on integrating over Ω Green's first identity for the Helmholtz equation,

$$\overline{v}\mathcal{L}u = \nabla \cdot [\overline{v}\nabla u] - \nabla u \cdot \overline{\nabla v} + k^2 u\overline{v}, \tag{1.26}$$

where $\mathcal{L}u := \Delta u + k^2 u$.

The new variational formulation (1.25) comes from integrating the following identity over Ω

$$\overline{\mathcal{M}_1 v}\mathcal{L}u + \mathcal{M}_2 u\overline{\mathcal{L}v} = \nabla \cdot \left[\overline{\mathcal{M}_1 v}\nabla u + \mathcal{M}_2 u\overline{\nabla v} + \mathbf{x}(k^2 u\overline{v} - \nabla u \cdot \overline{\nabla v})\right] \tag{1.27}$$
$$+ \left(d - 2 - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2)\right)\nabla u \cdot \overline{\nabla v} + \left(\alpha_1 + \alpha_2 - d + \mathrm{i}k(\beta_1 - \beta_2)\right)k^2 u\overline{v}.$$

where the multipliers $\mathcal{M}_j$ are defined by

$$\mathcal{M}_j v := \mathbf{x} \cdot \nabla v - \mathrm{i}k\beta_j v + \alpha_j v, \quad j = 1, 2, \tag{1.28}$$

and the real numbers β_j, α_j are chosen so as to ensure coercivity of the resulting sesquilinear form (the details are given in §3 and the derivation of (1.27) can be found in Lemma 2.2).

Although, to the authors' knowledge, the identity (1.27) has not been written down before, it arises naturally from existing ideas, which we now briefly explain.

Green's identity arises from multiplying $\mathcal{L}u$ by $\overline{v}$, for v an arbitrary test function, and Rellich-type identities arise from multiplying $\mathcal{L}u$ by a derivative of $\overline{v}$, most commonly $\mathbf{x} \cdot \overline{\nabla v}$. For the Laplace operator, multiplying by $\mathbf{x} \cdot \overline{\nabla v}$ yields the identity

$$(\mathbf{x} \cdot \overline{\nabla v})\Delta u = \nabla \cdot \left[(\mathbf{x} \cdot \overline{\nabla v})\nabla u\right] - \nabla u \cdot \overline{\nabla v} - \nabla u \cdot \left((\mathbf{x} \cdot \nabla)\overline{\nabla v}\right). \tag{1.29}$$

which is, in some sense, an analogue of Green's first identity for the Laplace operator (i.e., (1.26) with $k = 0$) with a different multiplier. However, (1.26) and (1.29) differ in two important respects: *(i)* the one non-divergence term on the right-hand side of (1.26) with $k = 0$ is single-signed (negative) when $u = v$, but, whereas the first non-divergence term on the right-hand side of (1.29) is single-signed when $u = v$, the second is not, *(ii)* the non-divergence terms on the right-hand side of (1.26) involve only first derivatives of u and v, whereas the second non-divergence term on the right-hand side of (1.29) involves second derivatives of v.

Because of these two considerations, we want to get rid of the final term on the right-hand side of (1.29). If we add to (1.29) the analogous expression with $\overline{v}$ and u swapped, we can use the identity

$$\nabla u \cdot \left((\mathbf{x} \cdot \nabla)\overline{\nabla v}\right) + \overline{\nabla v} \cdot \left((\mathbf{x} \cdot \nabla)\nabla u\right) = \nabla \cdot \left[\mathbf{x}\nabla u \cdot \overline{\nabla v}\right] - d\nabla u \cdot \overline{\nabla v} \tag{1.30}$$

to express the two undesirable terms as the sum of a divergence and a term with a constant sign when $u = v$. We thus arrive at

$$(\mathbf{x} \cdot \overline{\nabla v})\Delta u + (\mathbf{x} \cdot \nabla u)\overline{\Delta v} = \nabla \cdot \left[(\mathbf{x} \cdot \overline{\nabla v})\nabla u + (\mathbf{x} \cdot \nabla u)\overline{\nabla v} - \mathbf{x}\nabla u \cdot \overline{\nabla v}\right] + (d - 2)\nabla u \cdot \overline{\nabla v}, \tag{1.31}$$

which, in some sense, is an analogue of Green's second identity for the Laplacian,

$$\overline{v}\Delta u - u\overline{\Delta v} = \nabla \cdot [\overline{v}\nabla u - u\overline{\nabla v}], \tag{1.32}$$

since it involves both Δu and Δv. (The identity (1.31) appears as [48, Equation 2.5] and its generalisation from the Laplacian to a general 2nd order differential operator $\sum_{i,j} \partial_i(A_{ij}\partial_j)$ and from $\mathbf{x}$ to an arbitrary vector field is given in [43, Lemma 4.22].)

Having obtained the identity (1.31) involving the Laplace operator, it is then relatively straightforward to obtain the following identity involving the Helmholtz operator

$$(\mathbf{x} \cdot \overline{\nabla v})\mathcal{L}u + (\mathbf{x} \cdot \nabla u)\overline{\mathcal{L}v} = \nabla \cdot \left[(\mathbf{x} \cdot \overline{\nabla v})\nabla u + (\mathbf{x} \cdot \nabla u)\overline{\nabla v} + \mathbf{x}(k^2 u\overline{v} - \nabla u \cdot \overline{\nabla v})\right]$$
$$+ (d - 2)\nabla u \cdot \overline{\nabla v} - dk^2 u\overline{v} \tag{1.33}$$

(the details are in §2). This identity with $v = u$ was originally obtained by Rellich [60] and has been used extensively in the analysis of both the Laplace and Helmholtz equations (with suitable

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generalisations also used to study higher order elliptic PDEs). For example, Rellich himself introduced (1.33) with $v = u$ in order to obtain an expression for the eigenvalues of the Laplacian as an integral over $\partial\Omega$ (instead of the usual expression as an integral over Ω used in, e.g., the Rayleigh-Ritz method), and these identities have been used to further study eigenvalues of equations involving the Laplacian in, e.g., [59], [57], [36]. Rellich-type identites have been well-used by the harmonic analysis community (see, e.g., [41, Lemma 2.1.13 and Chapter 2 §10], [65, Lemma 2.2]), and more recently by the numerical analysis community to prove k-explicit bounds on the solution to (1.20) and related BVPs (see, e.g.,[44, Proposition 8.1.4], [18], [37], [14], [39])—some of this recent work is discussed in Remarks 3.6 and 4.7 below. (The recent review [12, §5.3] explains why Rellich-type identities can be used to do these things.)

Looking to use the identity (1.33) as the basis of a new variational formulation of the Helmholtz equation, we see that the non-divergence terms on the right-hand side of (1.33) involve only first derivatives of u and v, and each term has a sign when $u = v$, however for $d = 3$ the signs are opposite to one another, and for $d = 2$ we lose the $\nabla u \cdot \overline{\nabla v}$ term and thus have no hope of getting coercivity in a norm involving $|\nabla v|^2$. To remedy these facts, we add terms into the multiplier $\mathbf{x} \cdot \overline{\nabla v}$ to obtain the multiplier (1.28), and similarly for $\mathbf{x} \cdot \nabla u$, and then eventually obtain the identity (1.27). By choosing α_j and β_j appropriately we can ensure that both the non-divergence terms on the right-hand side of (1.27) are non-zero and have the same sign. This is not the only requirement for coercivity: we also need to control the term involving $\overline{\mathcal{L}v}$ on the left-hand side, as well as the divergence terms (which become integrals over $\partial\Omega$ when (1.27) is integrated over Ω), but these other requirements can ultimately also be achieved (see §3 for the details).

This idea of adding terms to the $\mathbf{x} \cdot \overline{\nabla v}$ multiplier (which can also be seen as taking certain linear combinations of the Rellich and Green multipliers) goes back to Morawetz (in [51] for the wave equation (1.2) and in [54] for the Helmholtz equation), and the identity (1.27) with $v = u$ essentially appears in [54] and [53] (see §2 for more details). These identities were used by Morawetz to prove bounds on solutions to the wave and Helmholtz equations, and have since been used in a variety of other contexts (see, e.g., [52], [19], [58]), including recently in a numerical analysis context by [62] and [63].

Why did we write the multiplier $\mathcal{M}_j$ in the particular form (1.28), with a k multiplying the constant β_j? The reason is, to obtain a coercive formulation with the smallest possible continuity constant C_c, the coefficient of iu in the multiplier must be proportional to k; thus it is natural to make this k-dependence explicit (so that β_j is then independent of k). When the multiplier $\mathcal{M}_j$ is used to prove bounds on solutions of the Helmholtz equation in exterior domains, $\mathcal{M}_j$ must be proportional to the first three terms in the asymptotic expansion of solutions satisfying the Sommerfeld radiation condition (see [54], [62, Remark 2.3]); thus explaining for this application why $\beta_j \sim 1$. Whereas this link with the radiation condition explains to a certain extent this choice of β_j for exterior problems, it is less clear why β_j should be taken to be independent of k to obtain a coercive formulation of the interior impedance problem (without going through the calculations). One possible explanation is that the multiplier should try to, in some sense, mimic the impedance boundary condition (1.20b). Indeed, in Section 3 below we consider the more general impedance boundary condition $\partial u/\partial n - \mathrm{i}k\vartheta u = g$, for an arbitrary function ϑ, and in this case the optimal β_j is independent of k, but depends on ϑ.

1.5 Outline of paper

In Section 2 we go through the details of deriving the main identity (1.27). In Section 3 we consider the interior impedance problem for the Helmholtz equation (1.20) and show how the identity (1.27) gives rise to the new coercive variational formulation (1.25). In Section 4 we consider the exterior sound-soft scattering problem for the Helmholtz equation (i.e. the Helmholtz equation posed in the exterior of a bounded domain with Dirichlet boundary conditions) and show that there exists a coercive variational formulation of this problem if the scatterer is star-shaped with respect to a ball. Section 5 begins to investigate some of the implications that the coercivity results have for potential discretisations of the Helmholtz equation. In Section 6 we discuss to what extent the geometric restriction of star-shapedness can be lifted from the new formulations of Sections 3 and 4. We conclude with some remarks in Section 7.

2   Morawetz- and Rellich-type identites

In Lemma 2.1 and Lemma 2.2 we prove identities (1.33) and (1.27), respectively.

Lemma 2.1 (Rellich-type identity)**.** *Let* $u, v \in C^2(D)$ *for some* $D \subset \mathbb{R}^d$, *and let* $\mathcal{L}v = \Delta v + k^2 v$ *where* $k \in \mathbb{R}$. *Then*

$$(\mathbf{x} \cdot \overline{\nabla v})\mathcal{L}u + (\mathbf{x} \cdot \nabla u)\overline{\mathcal{L}v} = \nabla \cdot \left[(\mathbf{x} \cdot \overline{\nabla v}) \nabla u + (\mathbf{x} \cdot \nabla u) \overline{\nabla v} + \mathbf{x}(k^2 u \overline{v} - \nabla u \cdot \overline{\nabla v}) \right]$$
$$+ (d-2)\nabla u \cdot \overline{\nabla v} - dk^2 u \overline{v}. \tag{2.1}$$

Proof. The identity (2.1) is the sum of (1.31) and k^2 times

$$(\mathbf{x} \cdot \overline{\nabla v})u + (\mathbf{x} \cdot \nabla u)\overline{v} = \nabla \cdot [\mathbf{x} \, u\overline{v}] - du\overline{v}. \tag{2.2}$$

The identity (1.31) arises from adding (1.29) to the same expression with u and $\overline{v}$ swapped, and using (1.30). To prove (1.29) and (1.30) expand the divergences on the right-hand sides using either the summation convention or the elementary vector calculus identities

$$\nabla \cdot [\mathbf{A}a] = a\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla a, \qquad \nabla(\mathbf{x} \cdot \nabla a) = \nabla a + (\mathbf{x} \cdot \nabla)\nabla a,$$

$$(\mathbf{x} \cdot \nabla)(\mathbf{A} \cdot \mathbf{B}) = \mathbf{B} \cdot \left((\mathbf{x} \cdot \nabla)\mathbf{A}\right) + \mathbf{A} \cdot \left((\mathbf{x} \cdot \nabla)\mathbf{B}\right),$$

which hold for any sufficiently differentiable scalar function a and vector fields $\mathbf{A}$, $\mathbf{B}$. (Note that (2.2) is the scalar analogue of (1.30).) $\qquad \square$

Rellich-type identities are most often used (and indeed derived) with $v = u$, i.e. one begins by multiplying $\mathcal{L}u$ by $\mathbf{x} \cdot \overline{\nabla u}$. In this case the "trick" (1.30) for getting rid of the undesirable term $\nabla u \cdot ((\mathbf{x} \cdot \nabla)\overline{\nabla u})$ becomes

$$2\Re\{\nabla u \cdot \left((\mathbf{x} \cdot \nabla)\overline{\nabla u}\right)\} = \nabla \cdot \left[\mathbf{x} |\nabla u|^2\right] - d|\nabla u|^2.$$

To use this we need to take the real part of the expression involving $\mathcal{L}u$, and this is the reason that Rellich identities for complex-valued functions always involve $2\Re\{(\mathbf{x} \cdot \overline{\nabla u})\,\mathcal{L}u\}$ (or this expression with a different vector field instead of $\mathbf{x}$).

Lemma 2.2 (Morawetz-type identity)**.** *Let* u, v *be as in Lemma 2.1 and define the operator* $\mathcal{M}_j$ *by*

$$\mathcal{M}_j v := \mathbf{x} \cdot \nabla v - \mathrm{i}k\beta_j v + \alpha_j v, \quad j = 1, 2, \tag{2.3}$$

where $\beta_j, \alpha_j \in \mathbb{R}$. *Then*

$$\overline{\mathcal{M}_1 v}\mathcal{L}u + \mathcal{M}_2 u\overline{\mathcal{L}v} = \nabla \cdot \left[\overline{\mathcal{M}_1 v}\,\nabla u + \mathcal{M}_2 u\,\overline{\nabla v} + \mathbf{x}(k^2 u\overline{v} - \nabla u \cdot \overline{\nabla v})\right] \tag{2.4}$$
$$+ \left(d - 2 - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2)\right)\nabla u \cdot \overline{\nabla v} + \left(\alpha_1 + \alpha_2 - d + \mathrm{i}k(\beta_1 - \beta_2)\right)k^2 u\overline{v}.$$

Proof. By Green's first identity

$$\overline{v}\mathcal{L}u = \nabla \cdot \left[\overline{v}\nabla u\right] - \nabla u \cdot \overline{\nabla v} + k^2 u\overline{v}, \tag{2.5}$$
$$u\overline{\mathcal{L}v} = \nabla \cdot \left[u\overline{\nabla v}\right] - \nabla u \cdot \overline{\nabla v} + k^2 u\overline{v}, \tag{2.6}$$

and then the identity (2.4) is the Rellich identity (2.1) plus $\mathrm{i}k\beta_1 + \alpha_1$ times (2.5), plus $-\mathrm{i}k\beta_2 + \alpha_2$ times (2.6). $\qquad \square$

Remark 2.3 (Relationship to other identities)**.** *If we let* $v = u$, $\beta_2 = \beta_1$, *and* $\alpha_2 = \alpha_1$ *in the identity (2.4) then we get*

$$2\Re\{\mathcal{M}_1 u\overline{\mathcal{L}u}\} = \nabla \cdot \left[2\Re\{\mathcal{M}_1 u\overline{\nabla u}\} + \mathbf{x}(k^2 |u|^2 - |\nabla u|^2)\right] + (d - 2 - 2\alpha_1)|\nabla u|^2 + (2\alpha_1 - d)k^2 |u|^2. \tag{2.7}$$

This identity is very similar to [54, Equation A.3] ([62, Equation 2.9]) except that the second term in that multiplier is $-\mathrm{i}k|\mathbf{x}|v$, *so the right-hand side then contains an extra term from differentiating* $|\mathbf{x}|$. *The analogous identity with* β_1 *and* α_1 *scalar functions of* $\mathbf{x}$, *and the vector* $\mathbf{x}$ *replaced by a more general vector field, essentially appears in [53, §I.2] ([53, Lemma 3] contains a particular case of this identity with* α_1 *and* β_1 *chosen to be certain functions of the vector field); the identity for general* α_1 *and* β_1 *appears in [63].*

In Sections 3 and 4 we need the identity (2.4) integrated over a Lipschitz domain Ω when u, v are in the space V defined by (1.21).

Lemma 2.4 (Integrated form of the main identity (2.4)). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with outward pointing normal $\mathbf{n}$. If $u, v \in V$, where V is defined by (1.21), then*

$$
\int_\Omega \left(\overline{\mathcal{M}_1 v} \mathcal{L} u + \mathcal{M}_2 u \overline{\mathcal{L} v} + \left(2 - d + \alpha_1 + \alpha_2 + \mathrm{i}k(\beta_1 - \beta_2) \right) \nabla u \cdot \overline{\nabla v} \right.
$$
$$
\left. + \left(d - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2) \right) k^2 u \overline{v} \right) \mathrm{d}\mathbf{x}
$$
$$
= \int_{\partial\Omega} \left(\overline{\mathcal{M}_1 v} \frac{\partial u}{\partial n} + \mathcal{M}_2 u \overline{\frac{\partial v}{\partial n}} + (\mathbf{x} \cdot \mathbf{n})(k^2 u \overline{v} - \nabla u \cdot \overline{\nabla v}) \right) \mathrm{d}s, \tag{2.8}
$$

where $\mathcal{L}u, \mathcal{L}v$ are as above, and $\mathcal{M}_j$ is defined by (2.3).

Proof. This is a consequence of the divergence theorem applied to the identity (2.4). The divergence theorem

$$
\int_\Omega \nabla \cdot \mathbf{F} \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s
$$

is valid when Ω is Lipschitz and $\mathbf{F} \in (C^1(\overline{\Omega}))^d$ [43, Theorem 3.34]. In Appendix A we prove that $\mathcal{D}(\overline{\Omega}) := \{U|_\Omega : U \in C^\infty(\mathbb{R}^d)\}$ is dense in V, and thus (2.8) holds for any $u, v \in V$. (Note that this density result is perhaps not immediately obvious due to the subtleties discussed in Remark 4.6 below.) $\qquad\square$

3 Interior Impedance Problem

Let Ω be a bounded Lipschitz domain with $L = \operatorname{diam}(\Omega)$, i.e. $L := \max_{\mathbf{x},\mathbf{y} \in \partial\Omega} |\mathbf{x} - \mathbf{y}|$. (For the definition of a Lipschitz domain see, e.g., [43, Definition 3.28], [61, Definition 2.2.7].) Note that we use the word domain to mean a *connected* open set (in common with [61]) not just an open set (as in [43]). For a Lipschitz domain the outward pointing normal vector $\mathbf{n}(\mathbf{x})$ is defined for almost every $\mathbf{x} \in \partial\Omega$ by Rademacher's theorem (see, e.g., [30, §3.1.2, Theorem 2]). In what follows, whenever we have an expression on $\partial\Omega$ we just write u instead of introducing any notation for the trace of u; we do this to prevent some of the expressions (e.g. (3.2)) from becoming over-complicated.

Definition 3.1 (Interior Impedance Problem). *Given $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and $\vartheta \in L^\infty(\partial\Omega)$ with ϑ real, independent of k and u, and such that*

$$
0 < \vartheta_* := \inf_{\mathbf{x} \in \partial\Omega} \vartheta(\mathbf{x}) \le \sup_{\mathbf{x} \in \partial\Omega} \vartheta(\mathbf{x}) =: \vartheta^* < \infty,
$$

find $u \in C^2(\Omega) \cap H^1(\Omega)$ such that

$$
\Delta u + k^2 u = -f \quad \text{in } \Omega, \tag{3.1a}
$$
$$
\frac{\partial u}{\partial n} - \mathrm{i}k\vartheta u = g \quad \text{on } \partial\Omega. \tag{3.1b}
$$

The solution to the interior impedance problem is unique. This is usually proved by applying Green's first identity (1.26) with $v = u$ and taking the imaginary part (equivalently considering $\Im a(u, u)$ with $a(\cdot, \cdot)$ given by (1.19)); the coercivity result below, however, gives an alternative proof of uniqueness in the space V defined by (1.21) (under the assumption that Ω is star-shaped with respect to a ball).

Recall that, for $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, $\partial u / \partial n$ is understood as an element of $H^{-1/2}(\partial\Omega)$ via Green's first identity (see, e.g., [43, Lemma 4.3], [12, Equation A.28]). For u a solution of the interior impedance problem, however, since $g \in L^2(\partial\Omega)$ we have that $\partial u / \partial n (= \mathrm{i}ku + g) \in L^2(\partial\Omega)$.

Note that if ϑ is chosen uniformly negative then all the results below follow in the same way (but we do not consider the cases where ϑ approaches zero, changes sign, or becomes unbounded).

Define the sesquilinear form $b : V \times V \to \mathbb{C}$ by

$$b(u,v) := \int_{\Omega} \Bigg(\big(2 - d + \alpha_1 + \alpha_2 + \mathrm{i}k(\beta_1 - \beta_2)\big)\nabla u \cdot \overline{\nabla v} + \big(d - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2)\big)k^2 u\overline{v}$$

$$+ \left(\mathcal{M}_2 u + \frac{A}{k^2}\mathcal{L}u\right)\overline{\mathcal{L}v}\Bigg)\mathrm{d}\mathbf{x} \tag{3.2}$$

$$- \int_{\partial\Omega} \left(\mathrm{i}k\vartheta u\,\overline{\mathcal{M}_1 v} + (\mathbf{x} \cdot \nabla_{\partial\Omega} u - \mathrm{i}k\beta_2 u + \alpha_2 u)\frac{\overline{\partial v}}{\partial n} + (\mathbf{x} \cdot \mathbf{n})\big(k^2 u\overline{v} - \nabla_{\partial\Omega} u \cdot \overline{\nabla_{\partial\Omega} v}\big) \right)\mathrm{d}s,$$

and the functional $G : V \to \mathbb{C}$ by

$$G(v) := \int_{\Omega} \left(\overline{\mathcal{M}_1 v} - \frac{A}{k^2}\overline{\mathcal{L}v}\right)f\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \overline{\mathcal{M}_1 v}\,g\,\mathrm{d}s, \tag{3.3}$$

where $\mathcal{M}_j$, $j = 1, 2$, are defined by (2.3), and $\alpha_1, \alpha_2, \beta_1, \beta_2$, and A are all arbitrary real parameters. If we take $\alpha_1 = \alpha_2 = (d-1)/2$, $\beta_1 = \beta_2 = \beta$, $A = 1/3$, and $\vartheta \equiv 1$ then (3.2) becomes (1.23).

Proposition 3.2 ($b(\cdot, \cdot)$ can be used to solve the interior impedance problem). *If u solves the interior impedance problem (Definition 3.1) then $u \in V$ (where V is defined by (1.21)), and*

$$b(u, v) = G(v) \quad \text{for all } v \in V, \tag{3.4}$$

where $b(\cdot, \cdot)$ is given by (3.2) and $G(\cdot)$ by (3.3).

Proof. For the solution of the interior impedance problem, u, to be in V we need to show that $\Delta u \in L^2(\Omega)$ and $\nabla u \in (L^2(\partial\Omega))^d$. From the PDE and boundary conditions (3.1) we have that $\Delta u = -k^2 u - f \in L^2(\Omega)$ and $\partial u / \partial n = \mathrm{i}ku + g \in L^2(\partial\Omega)$, and so, by a regularity result of Nečas [56, §5.2.1], [43, Theorem 4.24 (ii)], $\nabla_{\partial\Omega} u \in (L^2(\partial\Omega))^d$. Since ∇u is defined on $\partial\Omega$ by

$$\nabla u = \mathbf{n}\,\partial u/\partial n + \nabla_{\partial\Omega} u,$$

we have that $\nabla u \in (L^2(\partial\Omega))^d$ and thus $u \in V$.

Since u and v are both in V, the integrated identity (2.8) holds. From the definition of $\mathcal{M}_2 u$,

$$\mathcal{M}_2 u \frac{\overline{\partial v}}{\partial n} = (\mathbf{x} \cdot \mathbf{n})\frac{\partial u}{\partial n}\frac{\overline{\partial v}}{\partial n} + (\mathbf{x} \cdot \nabla_{\partial\Omega} u - \mathrm{i}k\beta_2 + \alpha_2 u)\frac{\overline{\partial v}}{\partial n}.$$

Substituting this last expression into (2.8), then using the PDE (3.1a) and boundary conditions (3.1b), and finally rearranging so that all the terms involving f and g are on one side and all the terms involving u are on the other we obtain

$$\int_{\Omega} \Bigg(\big(d - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2)\big)k^2 u\overline{v} + \big(2 - d + \alpha_1 + \alpha_2 + \mathrm{i}k(\beta_1 - \beta_2)\big)\nabla u \cdot \overline{\nabla v} + \mathcal{M}_2 u\overline{\mathcal{L}v}\Bigg)\mathrm{d}\mathbf{x}$$

$$- \int_{\partial\Omega} \left(\overline{\mathcal{M}_1 v}\,\mathrm{i}k\vartheta u + (\mathbf{x} \cdot \nabla_{\partial\Omega} u - \mathrm{i}k\beta_2 u + \alpha_2 u)\frac{\overline{\partial v}}{\partial n} + (\mathbf{x} \cdot \mathbf{n})\big(k^2 u\overline{v} - \nabla_{\partial\Omega} u \cdot \overline{\nabla_{\partial\Omega} v}\big) \right)\mathrm{d}s$$

$$= \int_{\Omega} \overline{\mathcal{M}_1 v}\,f\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \overline{\mathcal{M}_1 v}\,g\,\mathrm{d}s. \tag{3.5}$$

This is almost $b(u, v) = G(v)$ with $b(\cdot, \cdot)$ and $G(\cdot)$ defined by (3.2) and (3.3), respectively, but not quite. We need to add

$$\int_{\Omega} \frac{A}{k^2}\mathcal{L}u\overline{\mathcal{L}v}\,\mathrm{d}\mathbf{x}$$

to the left-hand side of (3.5) and

$$- \int_{\Omega} \frac{A}{k^2}f\,\overline{\mathcal{L}v}\,\mathrm{d}\mathbf{x}$$

to the right with $A \in \mathbb{R}$ arbitrary (these terms are equal to each other by the PDE (3.1a)). We add these terms because it turns out that $b(\cdot, \cdot)$ must contain a $\Delta u\overline{\Delta v}$ term to be coercive in the norm of V (1.22), and this term is not present in (3.5). (We could have just added a multiple of $\mathcal{L}u\overline{\Delta v}$, but we add $\mathcal{L}u\overline{\mathcal{L}v}$ to make $b(\cdot, \cdot)$ as symmetric as possible.) $\square$

As discussed above, the interior impedance problem has exactly one solution, which is in the space V. Theorem 3.4 below shows that the variational problem (3.4) has exactly one solution in V if Ω is star-shaped with respect to a ball; hence in this case the converse to Proposition 3.2 holds, namely, if u is a solution to (3.4) then u is a solution to the interior impedance problem.

Lemma 3.3 (Continuity of $b(\cdot,\cdot)$). *For all $u,v \in V$ and for all $k > 0$, the continuity bound*

$$|b(u,v)| \leq C_c \|u\|_V \|v\|_V$$

holds with

$$C_c := \sqrt{3} \max \begin{cases} |d - \alpha_1 - \alpha_2| + k|\beta_1 - \beta_2| + k|\beta_2| + |\alpha_2| + 2|A| + kL; \\ |2 - d + \alpha_1 + \alpha_2| + k|\beta_1 - \beta_2|; \\ 1 + \frac{\vartheta^*}{kL}(k|\beta_1| + |\alpha_1|); \\ 1 + \vartheta^* + \frac{1}{kL}(k|\beta_2| + |\alpha_2|). \end{cases}$$

If $\alpha_1 = \alpha_2 = (d-1)/2$, $\beta_1 = \beta_2 > 0$ and $A > 0$, then the above expression simplifies to

$$C_c = \sqrt{3} \max \left\{ \frac{d+1}{2} + k\beta_1 + 2A + kL; \ 1 + \vartheta^* + \frac{1+\vartheta^*}{kL}\left(k\beta_1 + \frac{d-1}{2}\right) \right\}.$$

In particular, if A and β_1 are independent of k (as we choose them to be below), then C_c grows linearly in k.

Proof. Define the vector $\mathbf{n}(u) \in \mathbb{R}^6$ by

$$\mathbf{n}(u) := \left\{ k\|v\|_\Omega \, ; \|\nabla v\|_\Omega \, ; k^{-1}\|\Delta v\|_\Omega \, ; L^{1/2}k\|v\|_{\partial\Omega} \, ; L^{1/2}\|\nabla_{\partial\Omega}v\|_{\partial\Omega} \, ; L^{1/2}\left\|\frac{\partial v}{\partial n}\right\|_{\partial\Omega} \right\},$$

so that $\|\mathbf{n}(u)\|_2 = \|u\|_V$, where $\|\cdot\|_2$ denotes the (Euclidean) 2-norm on $\mathbb{R}^6$. By Cauchy–Schwarz,

$$|b(u,v)| \leq |\mathbf{n}(u)^T \mathbf{M} \mathbf{n}(v)| \leq \|\mathbf{n}(u)\|_2 \|\mathbf{M}\|_2 \|\mathbf{n}(v)\|_2 = \|u\|_V \|\mathbf{M}\|_2 \|v\|_V$$

where $\mathbf{M}$ is a 6×6 block-diagonal matrix consisting of two 3×3 blocks $\mathbf{M}_1$ and $\mathbf{M}_2$. Thus $C_c \leq \|\mathbf{M}\|_2 = \max\{\|\mathbf{M}_1\|_2, \|\mathbf{M}_2\|_2\} \leq \sqrt{3}\max\{\|\mathbf{M}_1\|_1, \|\mathbf{M}_2\|_1\}$, and the assertion follows from the definition of $b(\cdot,\cdot)$, (3.2), which defines the coefficients of $\mathbf{M}$. $\qquad\square$

We now prove a more general version of the coercivity theorem stated in the introduction (Theorem 1.1).

Theorem 3.4 (Coercivity of $b(\cdot,\cdot)$). *Let $b(\cdot,\cdot)$ be defined by (3.2) and V defined by (1.21). Suppose that Ω is a Lipschitz domain with diameter L that is star-shaped with respect to a ball, i.e. there exists a $\gamma > 0$ such that*

$$\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq \gamma L, \tag{3.6}$$

for all $\mathbf{x} \in \partial\Omega$ for which $\mathbf{n}(\mathbf{x})$ is defined (see Remark 3.5 for the geometric significance of γ). If

$$\alpha_1 = \frac{d-1}{2}, \qquad \beta_1 \geq \frac{L}{2\vartheta_*}\left[1 + 4\frac{(\vartheta^*)^2}{\gamma} + \frac{\gamma}{2}\right], \qquad A = \frac{1}{3}, \tag{3.7}$$

then, for all $k > 0$, and for any $\alpha_2, \beta_2 \in \mathbb{R}$,

$$\Re b(v,v) \geq \frac{\gamma}{4}\|v\|_V^2 \quad \text{for all } v \in V.$$

Proof. The definition of $b(\cdot,\cdot)$ (3.2) implies that

$$2\Re b(v,v) = \int_\Omega \left(2(2 - d + \alpha_1 + \alpha_2)|\nabla v|^2 + 2(d - \alpha_1 - \alpha_2)k^2|v|^2 + \frac{2A}{k^2}|\mathcal{L}v|^2 + 2\Re\{\mathcal{M}_2 v\overline{\mathcal{L}v}\} \right) d\mathbf{x}$$

$$- 2\int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})(k^2|v|^2 - |\nabla_{\partial\Omega}v|^2)ds - 2\Re\int_{\partial\Omega} \left(ik\vartheta v\overline{\mathcal{M}_1 v} + (\mathbf{x}\cdot\nabla_{\partial\Omega}v - ik\beta_2 v + \alpha_2 v)\overline{\frac{\partial v}{\partial n}} \right) ds. \tag{3.8}$$

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Recall that (3.2) essentially came from the integrated identity (2.8). We now use this identity with $u = v$ to obtain an expression for the integral over Ω of $2\Re\{\mathcal{M}_2 v\overline{\mathcal{L}v}\}$, which appears on the right-hand side of (3.8). Indeed, (2.8) with $u = v$ implies that

$$\int_\Omega 2\Re\{\mathcal{M}_2 v\overline{\mathcal{L}v}\}\,\mathrm{d}\mathbf{x} = \int_\Omega \left((d - 2 - 2\alpha_2)|\nabla v|^2 + (2\alpha_2 - d)k^2|v|^2 \right)\mathrm{d}\mathbf{x}$$
$$+ \int_{\partial\Omega} \left((\mathbf{x}\cdot\mathbf{n})\left(\left|\frac{\partial v}{\partial n}\right|^2 + k^2|v|^2 - |\nabla_{\partial\Omega}v|^2 \right) + 2\Re\left\{ (\mathbf{x}\cdot\nabla_{\partial\Omega}v - \mathrm{i}k\beta_2 v + \alpha_2 v)\overline{\frac{\partial v}{\partial n}} \right\} \right)\mathrm{d}s.$$

Substituting this into (3.8) and using the definition of $\mathcal{M}_1$ (2.3) we find that

$$2\Re b(v, v) = \int_\Omega \left((2 - d + 2\alpha_1)|\nabla v|^2 + (d - 2\alpha_1)k^2|v|^2 + \frac{2A}{k^2}|\mathcal{L}v|^2 \right)\mathrm{d}\mathbf{x}$$
$$+ \int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})\left(\left|\frac{\partial v}{\partial n}\right|^2 + |\nabla_{\partial\Omega}v|^2 - k^2|v|^2 \right)\mathrm{d}s$$
$$- 2\Re\int_{\partial\Omega} \mathrm{i}k\vartheta v\left((\mathbf{x}\cdot\mathbf{n})\overline{\frac{\partial v}{\partial n}} + \mathbf{x}\cdot\overline{\nabla_{\partial\Omega}v} + \mathrm{i}k\beta_1\overline{v} + \alpha_1\overline{v} \right)\mathrm{d}s. \tag{3.9}$$

We first concentrate on the integral over Ω. Using both the triangle inequality and the inequality

$$2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2, \quad \text{for } a, b, \varepsilon > 0, \tag{3.10}$$

with $\varepsilon = 1$ we have that $|a|^2 \leq 2|b|^2 + 2|a + b|^2$, and thus $|a + b|^2 \geq \frac{1}{2}|a|^2 - |b|^2$. Using this with $a = \sqrt{2A}\Delta v/k$ and $b = \sqrt{2A}kv$, the integral over Ω in (3.9) is greater than or equal to

$$(2 - d + 2\alpha_1)\|\nabla v\|_{L^2(\Omega)}^2 + (d - 2\alpha_1 - 2A)k^2\|v\|_{L^2(\Omega)}^2 + \frac{A}{k^2}\|\Delta v\|_{L^2(\Omega)}^2.$$

If we choose $2\alpha_1 = d - 1$ and $A = 1/3$ then the coefficients of $\|\nabla v\|_{L^2(\Omega)}^2$, $k^2\|v\|_{L^2(\Omega)}^2$, and $\|\Delta v\|_{L^2(\Omega)}^2/k^2$ become 1, 1/3, and 1/3, respectively (other choices are possible, but the point is that we make each coefficient greater than zero.)

We now concentrate on the two integrals in (3.9) that are over $\partial\Omega$. Using the fact that $\gamma L \leq \mathbf{x}\cdot\mathbf{n} \leq L$ (since $|\mathbf{x}| \leq L$) the first integral is greater than or equal to

$$L\left(\gamma\left\|\frac{\partial v}{\partial n}\right\|_{L^2(\partial\Omega)}^2 + \gamma\|\nabla_{\partial\Omega}v\|_{L^2(\partial\Omega)}^2 - k^2\|v\|_{L^2(\partial\Omega)}^2 \right).$$

Because α_1 and θ are real the second integral over $\partial\Omega$ in (3.9) equals

$$-2\Re\int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})\overline{\frac{\partial v}{\partial n}}\mathrm{i}k\vartheta v\,\mathrm{d}s - 2\Re\int_{\partial\Omega} \mathbf{x}\cdot\overline{\nabla_{\partial\Omega}v}\,\mathrm{i}k\vartheta v\,\mathrm{d}s + 2k^2\beta_1\int_{\partial\Omega}\vartheta|v|^2\,\mathrm{d}s.$$

The final term in this last expression is $\geq 2\beta_1\theta_*k^2\|v\|_{L^2(\partial\Omega)}^2$. To deal with the first two terms we use the inequality (3.10) and the fact that $|\mathbf{x}\cdot\nabla_{\partial\Omega}v| \leq L|\nabla_{\partial\Omega}v|$ to obtain

$$2\Re\int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})\overline{\frac{\partial v}{\partial n}}\,\mathrm{i}k\vartheta v\,\mathrm{d}s \leq L\vartheta^*\left(\frac{1}{\varepsilon_1}\left\|\frac{\partial v}{\partial n}\right\|_{L^2(\partial\Omega)}^2 + \varepsilon_1 k^2\|v\|_{L^2(\partial\Omega)}^2 \right)$$

and

$$2\Re\int_{\partial\Omega} \mathbf{x}\cdot\overline{\nabla_{\partial\Omega}v}\,\mathrm{i}k\vartheta v\,\mathrm{d}s \leq L\vartheta^*\left(\frac{1}{\varepsilon_2}\|\nabla_{\partial\Omega}v\|_{L^2(\partial\Omega)}^2 + \varepsilon_2 k^2\|v\|_{L^2(\partial\Omega)}^2 \right)$$

for any $\varepsilon_1, \varepsilon_2 > 0$.

Putting everything together results in the inequality

$$2\Re b(v, v) \geq \|\nabla v\|_{L^2(\Omega)}^2 + \frac{1}{3}k^2\|v\|_{L^2(\Omega)}^2 + \frac{1}{3k^2}\|\mathcal{L}v\|_{L^2(\Omega)}^2 + L\left(\gamma - \frac{\vartheta^*}{\varepsilon_1} \right)\left\|\frac{\partial v}{\partial n}\right\|_{L^2(\partial\Omega)}^2$$

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$$+ L\left(\gamma - \frac{\vartheta^*}{\varepsilon_2}\right) \|\nabla_{\partial\Omega} v\|^2_{L^2(\partial\Omega)} + L\left(\frac{2\beta_1 \vartheta_*}{L} - 1 - \vartheta^* \varepsilon_1 - \vartheta^* \varepsilon_2\right) k^2 \|v\|^2_{L^2(\partial\Omega)}. \quad (3.11)$$

If we choose $\varepsilon_1 = \varepsilon_2 = 2\vartheta^*/\gamma$ then the norms on $\partial\Omega$ in (3.11) become

$$\frac{\gamma L}{2}\left(\left\|\frac{\partial v}{\partial n}\right\|^2_{L^2(\partial\Omega)} + \|\nabla_{\partial\Omega} v\|^2_{L^2(\partial\Omega)}\right) + L\left(\frac{2\beta_1 \vartheta_*}{L} - 1 - 4\frac{(\vartheta^*)^2}{\gamma}\right) k^2 \|v\|^2_{L^2(\partial\Omega)}.$$

Then, if we choose β_1 as in (3.7) these terms are greater than or equal to

$$\frac{\gamma L}{2}\left(\left\|\frac{\partial v}{\partial n}\right\|^2_{L^2(\partial\Omega)} + \|\nabla_{\partial\Omega} v\|^2_{L^2(\partial\Omega)} + k^2 \|v\|^2_{L^2(\partial\Omega)}\right).$$

Hence, the inequality (3.11) becomes

$$2\Re b(v,v) \geq \|\nabla v\|^2_{L^2(\Omega)} + \frac{1}{3}k^2 \|v\|^2_{L^2(\Omega)} + \frac{1}{3k^2} \|\mathcal{L}v\|^2_{L^2(\Omega)}$$
$$+ \frac{\gamma L}{2}\left(\left\|\frac{\partial v}{\partial n}\right\|^2_{L^2(\partial\Omega)} + \|\nabla_{\partial\Omega} v\|^2_{L^2(\partial\Omega)} + k^2 \|v\|^2_{L^2(\partial\Omega)}\right),$$

and so $b(\cdot, \cdot)$ is coercive with

$$\alpha = \frac{1}{2}\min\left\{\frac{1}{3}; \frac{\gamma}{2}\right\} = \frac{\gamma}{4}$$

(since $\gamma \leq 1/2$ by Remark 3.5 below). $\qquad\square$

Note that α_2 and β_2 do not play any role in the proof that $b(\cdot, \cdot)$ is coercive.

Remark 3.5 (Geometrical significance of γ in the star-shapedness condition (3.6))**.** *The standard definition of star-shapedness is that Ω is star-shaped with respect to a point $\mathbf{x}_0$ if, whenever $\mathbf{x} \in \Omega$, the segment $(\mathbf{x}_0, \mathbf{x}) \subset \Omega$. Furthermore, Ω is star-shaped with respect to the ball $B_a(\mathbf{x}_0)$ if it is star-shaped with respect to every point in $B_a(\mathbf{x}_0)$.*

If Ω is Lipschitz (and so has a normal vector at almost every point on the boundary) then

Ω *is star-shaped with respect to $B_a(\mathbf{x}_0) \Leftrightarrow \mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq a$ for all $\mathbf{x} \in \partial\Omega$ for which $\mathbf{n}(\mathbf{x})$ is defined;*

for a proof see [49, Lemma 5.4.1] or [39, Lemma 3.1].

Remark 3.6 (Bounding the solution of the BVP)**.** *Combining Theorem 3.4, the fact that*

$$\|G\|_{V'} \leq \sqrt{3}\max\left\{1; \frac{1}{kL}(k|\beta_1| + |\alpha_1| + |A|)\right\}\left(L^2 \|f\|^2_{L^2(\Omega)} + L \|g\|^2_{L^2(\partial\Omega)}\right)^{1/2},$$

and the consequence of coercivity (1.9), we obtain the bound

$$\|u\|_V \leq \frac{4\sqrt{3}}{\gamma}\left(1 + \frac{\beta_1}{L} + \frac{d}{2kL}\right)\left(L^2 \|f\|^2_{L^2(\Omega)} + L \|g\|^2_{L^2(\partial\Omega)}\right)^{1/2}, \quad (3.12)$$

for all $k > 0$ (under the condition (3.7) on α_1, β_1 and A). Bounds identical to (3.12) in their k-dependence were proved for $d = 2$ in [44, Proposition 8.1.4] and for $d = 3$ in [18, Theorem 1], essentially using the identity (2.7) with $\beta_1 = 0$; see [12, §5.3.2] for more explanation. It is interesting to note that taking β_1 to be non-zero in the multiplier $\mathcal{M}_1 u$ does not help in proving the bounds on the solution, but is crucial for the proof that $b(\cdot, \cdot)$ is coercive (since we need to take β_1 large enough to get coercivity).

Remark 3.7 (Relationship of the space V to $H^{3/2}(\Omega)$)**.** *We now outline how to prove the facts mentioned in §1.2 that (i) $V \subset H^{3/2}(\Omega)$, and (ii) if $v \in C^2(\Omega)$ is such that $\Delta v + k^2 v = 0$, then $v \in H^{3/2}(\Omega)$ implies that $v \in V$. The statement (i) follows from expressing u via the Green's integral representation that involves the fundamental solution of the Laplacian [43, Theorem 7.5] and then using mapping properties of the Newtonian potential [43, Theorem 6.1] and the single- and double-layer potential operators [12, Theorem 2.15]. The statement (ii) follows from [12, Corollary A.8]).*

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Remark 3.8 (Why the norm in V is scaled with k and L). *If v is a plane wave solution to the Helmholtz equation, i.e. $v = \exp(ik\mathbf{x} \cdot \widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $\|\widehat{\mathbf{a}}\| = 1$, then each of the terms in $\|v\|_V^2$ is proportional to $k^2 L^d$. Similarly, if $f = \Delta v + k^2 v$ and $g = \partial v/\partial n - ik\vartheta v$, then the factor involving f and g in* (3.12) *is also proportional to $k^2 L^d$.*

4   Sound-soft scattering problem

Let Ω_D be a bounded Lipschitz domain (with the subscript D because we are going to consider Dirichlet boundary conditions) and set $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_D}$. Let $H^1_{\text{loc}}(\Omega_+)$ denote the set of functions, v, such that v is locally integrable on Ω_+ and $\psi v \in H^1(\Omega_+)$ for every compactly supported $\psi \in C^\infty(\overline{\Omega_+}) := \{\psi|_{\overline{\Omega_+}} : \psi \in C^\infty(\mathbb{R}^d)\}$.

Definition 4.1 (Sound-soft scattering problem). *Given an incident plane wave $u^I(\mathbf{x}) = \exp(ik\mathbf{x} \cdot \widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $\|\widehat{\mathbf{a}}\| = 1$, find $u^S \in C^2(\Omega_+) \cap H^1_{\text{loc}}(\Omega_+)$ such that the total field $u^T := u^I + u^S$ satisfies*

$$\Delta u^T + k^2 u^T = 0 \quad in \ \Omega_+,$$
$$u^T = 0 \quad on \ \partial\Omega_+,$$

and u^S satisfies the Sommerfeld radiation condition

$$\frac{\partial u^S}{\partial r}(\mathbf{x}) - iku^S(\mathbf{x}) = o\left(\frac{1}{r^{(d-1)/2}}\right),$$

as $r := |\mathbf{x}| \to \infty$, uniformly in $\widehat{\mathbf{x}} := \mathbf{x}/r$.

(Note that we can replace the incident plane wave u^I by a more general incident field; see, e.g. [12, Definition 2.11].)

Since Ω_+ is unbounded, standard finite element methods (FEMs) cannot be applied to solve the sound-soft scattering problem. One way around this is to truncate Ω_+, i.e. introduce an artificial boundary Γ_R, and impose a boundary condition on Γ_R that approximates the radiation condition. The design of appropriate boundary conditions has been, and still is, the subject of much research (see [40, Chapter 3] for an introduction), but the simplest option is just to impose an impedance boundary condition on Γ_R. We thus consider the following BVP:

Definition 4.2 (Truncated sound-soft scattering problem). *Given Ω_R and Ω_D bounded Lipschitz domains such that $\Omega_D \subset \Omega_R \subset \mathbb{R}^d$ with $d(\Omega_D, \partial\Omega_R) > 0$, let $\Gamma_R := \partial\Omega_R$, $\Gamma_D := \partial\Omega_D$, and $\Omega := \Omega_R \setminus \overline{\Omega_D}$ (so $\partial\Omega = \Gamma_R \cup \Gamma_D$ and $\Gamma_R \cap \Gamma_D = \emptyset$). Given $f \in L^2(\Omega_R)$, $g_R \in L^2(\Gamma_R)$, $g_D \in H^1(\Gamma_D)$, and $\vartheta \in L^\infty(\partial\Omega_R)$ with ϑ real, independent of k and u, and such that*

$$0 < \vartheta_* := \inf_{\mathbf{x} \in \partial\Omega_R} \vartheta(\mathbf{x}) \leq \sup_{\mathbf{x} \in \partial\Omega_R} \vartheta(\mathbf{x}) =: \vartheta^* < \infty,$$

find $u \in C^2(\Omega) \cap H^1(\Omega)$ such that

$$\Delta u + k^2 u = -f \quad in \ \Omega, \tag{4.1a}$$

$$\frac{\partial u}{\partial n} - i\vartheta k u = g_R \quad on \ \Gamma_R, \tag{4.1b}$$

$$u = g_D \quad on \ \Gamma_D. \tag{4.1c}$$

If we set $f = 0$, $\vartheta = 1$, $g_R = 0$, and $g_D = -u^I|_{\Gamma_D}$ then the solution to the BVP in Definition 4.2 is an approximation to u^S in Definition 4.1. The simplest choice for Ω_R is just $B_R(\mathbf{0})$ where R is taken large enough so that the ball includes Ω_D.

With Ω defined as in Definition 4.2, define the Hilbert space V by (1.21) with associated norm

$$\|v\|_V^2 := \|\nabla v\|_{L^2(\Omega)}^2 + k^2 \|v\|_{L^2(\Omega)}^2 + k^{-2} \|\Delta v\|_{L^2(\Omega)}^2$$
$$+ L\left(k^2 \|v\|_{L^2(\Gamma_R)}^2 + \|\nabla v\|_{L^2(\Gamma_R)}^2 + k^2 \|v\|_{L^2(\Gamma_D)}^2 + \|\nabla v\|_{L^2(\Gamma_D)}^2\right),$$

where, $L = \text{diam}(\Omega)$. (Note that we could have formulated the truncated sound-soft scattering problem in terms of the total field u^T, with the zero Dirichlet boundary condition on Γ_D imposed in the space V. We choose not to do this for technical reasons explained below in Remark 4.6.)

Let $\mathbf{n}_R$ be the outward-pointing normal to Ω_R, and let $\mathbf{n}_D$ be the outward-pointing normal to Ω_D (so $\mathbf{n}_D$ is the *inward* pointing normal to Ω on Γ_D); we use the convention that on Γ_D the normal derivative is $\partial v/\partial n = \mathbf{n}_D \cdot \nabla v$, and similarly on Γ_R.

Define the sesquilinear form $b : V \times V \to \mathbb{C}$ by

$$
\begin{aligned}
b(u,v) := \int_\Omega \Bigg(& \left(2 - d + \alpha_1 + \alpha_2 + \mathrm{i}k(\beta_1 - \beta_2)\right) \nabla u \cdot \overline{\nabla v} + \left(d - \alpha_1 - \alpha_2 - \mathrm{i}k(\beta_1 - \beta_2)\right) k^2 u \overline{v} \\
& + \left(\mathcal{M}_2 u + \frac{A_1}{k^2}\mathcal{L}u\right)\overline{\mathcal{L}v}\Bigg)\mathrm{d}\mathbf{x} + \int_{\Gamma_D} \left(\frac{\partial u}{\partial n}\overline{\mathcal{M}_1 v} + LA_2 k^2 u\overline{v}\right) \mathrm{d}s \\
& - \int_{\Gamma_R} \left(\mathrm{i}k\vartheta u\overline{\mathcal{M}_1 v} + (\mathbf{x}\cdot\nabla_{\Gamma_R}u - \mathrm{i}k\beta_2 u + \alpha_2 u)\overline{\frac{\partial v}{\partial n}} + (\mathbf{x}\cdot\mathbf{n})\left(k^2 u\overline{v} - \nabla_{\Gamma_R}u\cdot\overline{\nabla_{\Gamma_R}v}\right)\right)\mathrm{d}s,
\end{aligned}
\tag{4.2}
$$

and functional $G : V \to \mathbb{C}$ by

$$
\begin{aligned}
G(v) := & \int_\Omega \left(\overline{\mathcal{M}_1 v} - \frac{A_1}{k^2}\overline{\mathcal{L}v}\right) f\,\mathrm{d}\mathbf{x} + \int_{\Gamma_R} \overline{\mathcal{M}_1 v}\, g_R\,\mathrm{d}s \\
& - \int_{\Gamma_D} \left((\mathbf{x}\cdot\nabla_{\Gamma_D}g_D - \mathrm{i}k\beta_2 g_D + \alpha_2 g_D)\overline{\frac{\partial v}{\partial n}} + (\mathbf{x}\cdot\mathbf{n})(k^2 g_D\overline{v} - \nabla_{\Gamma_D}g_D\cdot\overline{\nabla_{\Gamma_D}v}) - LA_2 k^2 g_D\overline{v}\right)\mathrm{d}s,
\end{aligned}
\tag{4.3}
$$

where $\mathcal{M}_j$, $j = 1,2$, are defined by (2.3), $\alpha_1, \alpha_2, \beta_1, \beta_2, A_1$, and A_2 are all arbitrary real parameters, and ∇_{Γ_D} and ∇_{Γ_R} are the surface gradients on Γ_D and Γ_R, respectively. Note that (4.2) and (4.3) are the same as the $b(\cdot,\cdot)$ and $G(\cdot)$ for the interior impedance problem, (3.2) and (3.3) respectively, (identifying Γ_R with $\partial\Omega$) except for extra terms on Γ_D.

We now prove the analogues of Proposition 3.2, Lemma 3.3, and Theorem 3.4.

Proposition 4.3 ($b(\cdot,\cdot)$ can be used to solve the truncated sound-soft scattering problem). *If u solves the truncated sound-soft scattering problem (Definition 4.2) then $u \in V$ (where V is defined by (1.21)), and*
$$b(u,v) = G(v) \quad \text{for all } v \in V,$$
where $b(\cdot,\cdot)$ is given by (4.2) and $G(\cdot)$ by (4.3).

Proof. This is very similar to the proof of Proposition 3.2. The fact that $f \in L^2(\Omega)$ implies that $\Delta u \in L^2(\Omega)$, the fact that $g_R \in L^2(\Gamma_R)$ implies that $\partial u/\partial n \in (L^2(\Gamma_R))^d$, and the fact that $g_D \in H^1(\Gamma_D)$ implies that $u \in H^1(\Gamma_D)$. To show that $u \in V$, we again use the results of Nečas [56, §5.1.2, §5.2.1], [43, Theorem 4.2.4], which show that, for $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, the conditions $u \in H^1(\partial\Omega)$ and $\partial u/\partial n \in L^2(\partial\Omega)$ are equivalent. However, the fact that we have different boundary conditions on Γ_D and Γ_R means that to prove that $\nabla u \in (L^2(\partial\Omega))^d$ we need to introduce a smooth boundary, Γ^*, between Γ_D and Γ_R, and apply the Nečas result first between Γ^* and Γ_D, and then between Γ^* and Γ_R (using interior H^2-regularity of the Laplacian [29, §6.3.1, Theorem 1] and the trace theorem [43, Theorem 3.38] to get $\nabla u \in L^2(\Gamma^*)^d$).

To obtain $b(u,v) = G(v)$ we apply the integrated identity (2.8) in Ω and use the boundary conditions (4.1b) and (4.1c). As in the interior case, we add on a multiple of $\mathcal{L}u\overline{\mathcal{L}v}$ to both sides of the identity (involving the constant A_1), but now we also add on a multiple of $u\overline{v}$ on Γ_D (involving the constant A_2); this turns out to be necessary for coercivity. $\qquad\square$

Lemma 4.4 (Continuity of $b(\cdot,\cdot)$). *If α_j, β_j, and A_j, $j = 1,2$, are all independent of k, then*
$$|b(u,v)| \le C'_c \|u\|_V \|v\|_V$$
for all $u,v \in V$ and for all $k > 0$ where
$$C'_c := \max\left\{C_c;\ A_2 + \frac{1}{kL}(k|\beta_1| + |\alpha_1|)\right\},$$
and where C_c is as in Lemma 3.3 with A replaced by A_1.

Proof. This is almost identical to that of Lemma 3.3, except that now there are traces on both Γ_D and Γ_R, and so the vector $\mathbf{n}(u) \in \mathbb{R}^9$. $\qquad\square$

Theorem 4.5 (Coercivity of $b(\cdot, \cdot)$). *Let $b(\cdot, \cdot)$ be defined by (4.2) and V defined by (1.21). Suppose that Ω_R is a Lipschitz domain with diameter L that is star-shaped with respect to a ball, i.e. there exists a $\gamma_R > 0$ such that*

$$\mathbf{x} \cdot \mathbf{n}_R(\mathbf{x}) \geq \gamma_R L,$$

for all $\mathbf{x} \in \Gamma_R$ for which $\mathbf{n}_R(\mathbf{x})$ is defined, and suppose that Ω_D is Lipschitz and star-shaped with respect to a ball with the same centre as the previous one, i.e. there exists a $\gamma_D > 0$ such that

$$\mathbf{x} \cdot \mathbf{n}_D(\mathbf{x}) \geq \gamma_D L,$$

for all $\mathbf{x} \in \Gamma_D$ for which $\mathbf{n}_D(\mathbf{x})$ is defined. If

$$\alpha_1 = \alpha_2 = \frac{d-1}{2}, \quad \beta_1 = \beta_2 \geq \frac{L}{2\vartheta_*}\left[1 + 4\frac{(\vartheta^*)^2}{\gamma_R} + \frac{\gamma_R}{2}\right], \quad A_1 = \frac{1}{3}, \quad A_2 = 1, \qquad (4.4)$$

then, for all $k > 0$,

$$\Re b(v, v) \geq \alpha \|v\|_V^2 \quad \text{for all } v \in V,$$

with

$$\alpha = \frac{1}{2}\min\left\{\frac{\gamma_R}{2}; \gamma_D\right\}. \qquad (4.5)$$

Proof. This follows the same steps as the proof of Theorem 3.4, with some small differences. As before, the definition of $b(\cdot, \cdot)$ implies that

$$2\Re b(v, v) = \int_\Omega \left(2(2 - d + \alpha_1 + \alpha_2)|\nabla v|^2 + 2(d - \alpha_1 - \alpha_2)k^2|v|^2 + \frac{2A_1}{k^2}|\mathcal{L}v|^2 + 2\Re\{\mathcal{M}_2 v \overline{\mathcal{L}v}\}\right)\mathrm{d}\mathbf{x}$$

$$- 2\int_{\Gamma_R}(\mathbf{x} \cdot \mathbf{n}_R)(k^2|v|^2 - |\nabla_{\Gamma_R}v|^2)\mathrm{d}s - 2\Re\int_{\Gamma_R}\left(ik\vartheta v \overline{\mathcal{M}_1 v} + (\mathbf{x} \cdot \nabla_{\Gamma_R}v - ik\beta_2 v + \alpha_2 v)\overline{\frac{\partial v}{\partial n}}\right)\mathrm{d}s$$

$$+ \int_{\Gamma_D}\left(2\Re\left\{\overline{\mathcal{M}_1 v}\frac{\partial v}{\partial n}\right\} + 2LA_2 k^2|v|^2\right)\mathrm{d}s. \qquad (4.6)$$

As in the proof of Theorem 3.4 we use the identity (2.8) with $u = v$ to obtain the following expression for the integral over Ω of $2\Re\{\overline{\mathcal{M}_2 v}\mathcal{L}v\}$,

$$\int_\Omega 2\Re\{\mathcal{M}_2 v \overline{\mathcal{L}v}\}\,\mathrm{d}\mathbf{x} = \int_\Omega\left((d - 2 - 2\alpha_2)|\nabla v|^2 + (2\alpha_2 - d)k^2|v|^2\right)\mathrm{d}\mathbf{x}$$

$$+ \int_{\Gamma_R}\left((\mathbf{x} \cdot \mathbf{n}_R)\left(\left|\frac{\partial v}{\partial n}\right|^2 + k^2|v|^2 - |\nabla_{\Gamma_R}v|^2\right) + 2\Re\{\mathbf{x} \cdot \nabla_{\Gamma_R}v - ik\beta_2 v + \alpha_2 v\}\overline{\frac{\partial v}{\partial n}}\right)\mathrm{d}s$$

$$- \int_{\Gamma_D}\left(2\Re\left\{\overline{\mathcal{M}_2 v}\frac{\partial v}{\partial n}\right\} + (\mathbf{x} \cdot \mathbf{n}_D)(k^2|v|^2 - |\nabla v|^2)\right)\mathrm{d}s, \qquad (4.7)$$

(recall that $\mathbf{n}_D$ points into Ω and $\mathbf{n}_R$ points outwards). We substitute (4.7) into (4.6) and take $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, so that $\mathcal{M}_1 v = \mathcal{M}_2 v$ and thus the corresponding terms on Γ_D cancel. We end up with

$$2\Re b(v, v) = \int_\Omega\left((2 - d + 2\alpha_1)|\nabla v|^2 + (d - 2\alpha_1)k^2|v|^2 + \frac{2A_1}{k^2}|\mathcal{L}v|^2\right)\mathrm{d}\mathbf{x}$$

$$+ \int_{\Gamma_R}(\mathbf{x} \cdot \mathbf{n}_R)\left(\left|\frac{\partial v}{\partial n}\right|^2 + |\nabla_{\Gamma_R}v|^2 - k^2|v|^2\right)\mathrm{d}s$$

$$- 2\Re\int_{\Gamma_R}\left((\mathbf{x} \cdot \mathbf{n}_R)\overline{\frac{\partial v}{\partial n}} + \mathbf{x} \cdot \overline{\nabla_{\Gamma_R}v} + ik\beta_1\overline{v} + \alpha_1\overline{v}\right)ik\vartheta v\,\mathrm{d}s$$

$$+ \int_{\Gamma_D}\left(2LA_2 k^2|v|^2 + (\mathbf{x} \cdot \mathbf{n}_D)\left(\left|\frac{\partial v}{\partial n}\right|^2 + |\nabla_{\Gamma_D}v|^2 - k^2|v|^2\right)\right)\mathrm{d}s. \qquad (4.8)$$

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The terms on Γ_R and in Ω are dealt with in exactly the same way as in the proof of Theorem 3.4. The terms on Γ_D are greater than or equal to

$$\gamma_D L \left(\|\nabla_{\Gamma_D} v\|_{L^2(\Gamma_D)}^2 + \left\| \frac{\partial v}{\partial n} \right\|_{L^2(\Gamma_D)}^2 \right) + L(2A_2 - 1)k^2 \|v\|_{L^2(\Gamma_D)}^2 .$$

Thus, choosing $A_2 = 1$ and remembering that $\gamma_D \leq 1/2$ (by Remark 3.5) we get coercivity with constant (4.5). $\qquad\square$

Remark 4.6 (Technical considerations)**.** *We formulated the truncated Dirichlet scattering problem in terms of the scattered field, with a Dirichlet boundary condition on Γ_D imposed weakly. Instead, we could have formulated the problem in terms of the total field, and imposed the Dirichlet boundary condition in a strong form using the space*

$$V_0 := \left\{ v \in H^1(\Omega), \ v|_{\Gamma_D} = 0, \ \Delta v \in L^2(\Omega), \ \nabla v \in (L^2(\partial\Omega))^d \right\}.$$

It turns out that the analogous variational formulation is also coercive and continuous on this space, but there is a subtle disadvantage: for a general Lipschitz domain D, $H^2(D) \cap H_0^1(D)$ is not dense in $H_0^1(D, \Delta) := \{v \in H^1(D), \ v|_{\partial D} = 0, \ \Delta v \in L^2(D)\}$ if D has a reentrant corner. (The fact that, whenever D has reentrant corners, $H_0^1(D, \Delta) \setminus H^2(D)$ is non-empty is well-known [33, Lemma 4.4.3.5], [35, Page 576]. The fact that $H_0^1(D, \Delta) \cap H^2(D)$ is closed in $H_0^1(D, \Delta)$ follows from a corollary of the open mapping theorem [61, Corollary 2.1.8] using the facts that i) $\Delta^{-1} : L^2(D) \to H_0^1(D, \Delta)$ is a continuous bijection and ii) $\Delta : H_0^1(D, \Delta) \cap H^2(D) \to L^2(D)$ is continuous, which both follow from the bounds $\|\Delta u\|_{L^2(D)} \leq \|u\|_{H_0^1(D, \Delta)} \leq C \|\Delta u\|_{L^2(D)}$ for all $u \in H_0^1(D, \Delta)$.) For the truncated scattering problem this implies that if Ω_D has a corner then $H^2(\Omega) \cap V_0$ is not dense in V_0. However, Lemma 5.1 below implies that any conforming finite element method in V_0 consists of functions that are in $H^2(\Omega)$, and thus these functions are not able to approximate general Helmholtz solutions in V_0. This is analogous to the situation encountered in the context of the time-harmonic Maxwell's equations where $\mathbf{X_N} \cap (H^1(D))^3$ is not dense in $\mathbf{X_N}$ for D a non-convex polyhedron, where

$$\mathbf{X_N} := \left\{ \mathbf{u} \in (L^2(D))^3 \ : \ \nabla \times \mathbf{u} \in (L^2(D))^3, \ \nabla \cdot \mathbf{u} \in L^2(D), \ \mathbf{u} \times \mathbf{n} = \mathbf{0} \ on \ \partial D \right\},$$

see [50, Lemma 3.56]. This is a well-known fact since it prevents H^1-conforming finite element approximations from converging to singular solutions.

Remark 4.7 (Bounding the solution of the BVP)**.** *In analogy with the case of the interior problem discussed in Remark 3.6, the coercivity result Theorem 4.5, together with (1.9), gives the following stability bound on the solution of the BVP:*

$$\|u\|_V \leq \frac{4\sqrt{3} \left(2 + \frac{\beta_1}{L} + \frac{d}{2kL} \right)}{\min\{\gamma_R; \ 2\gamma_D\}} \tag{4.9}$$
$$\cdot \left(L^2 \|f\|_{L^2(\Omega)}^2 + L \|g_R\|_{L^2(\Gamma_R)}^2 + k^2 L \|g_D\|_{L^2(\Gamma_D)}^2 + L \|\nabla_{\Gamma_D} g_D\|_{L^2(\Gamma_D)}^2 \right)^{1/2}$$

for all $k > 0$. A bound identical to this one in its k-dependence was obtained in [37, Proposition 3.3]; although only the case $g_D \equiv 0$ was considered there, the same method can be used to obtain a bound for the case of non-homogeneous Dirichlet boundary conditions.

5 Implications for numerical methods

On the one hand, the primary aim of this paper is to introduce the new coercive formulations of Sections 3 and 4 as results about the Helmholtz equation itself, independent of potential discretisations. On the other hand, one can almost immediately obtain results about possible Galerkin discretisations from the property of coercivity, and so it would seem a shame not to do this. In this section, therefore, we briefly begin to investigate potential discretisations of the new variational formulation. The actual implementation of these discretisations, and a more thorough study of their properties, will be presented elsewhere.

5.1 Conforming finite element methods

We first show that, for Ω a general bounded Lipschitz domain, the requirement that $\Delta v \in L^2(\Omega)$ in the space V (defined by (1.21)) means that any conforming finite element method in this space must use C^1-elements.

Lemma 5.1 (C^1-conformity). *Let $\mathcal{T}$ be a triangulation of Ω (in the sense of [16, Page 61]). If $v \in V$ is also in $C^2(\overline{K})$ for each $K \in \mathcal{T}$, then $v \in C^1(\overline{\Omega}) \cap H^2(\Omega)$.*

Proof. The fact that a piecewise C^k function belongs to $H^k(\Omega)$ if and only if it belongs to $C^{k-1}(\overline{\Omega})$ is well-known (see, e.g., [7, Theorem II.5.2, Page 62] or [16, Theorems 5.1 and 30.1, Pages 62 and 210]); the proof of this lemma is extremely similar to the proof of the forward implication. Since $V \subset H^1(\Omega)$, for any $v \in V$ that is piecewise C^2 we have that $v \in C(\overline{\Omega})$; thus we only need to show that $\nabla v \in C(\overline{\Omega})$, and then $v \in H^2(\Omega)$ follows from the result mentioned above.

The fact that $\Delta v \in L^2(\Omega)$ implies that, for any $\phi \in C^\infty_{\text{comp}}(\Omega) := \{v \in C^\infty(\Omega), \text{supp } v \subset\subset \Omega\}$,

$$\int_\Omega \phi \, \Delta v \, \mathrm{d}\mathbf{x} = \int_\Omega v \, \Delta \phi \, \mathrm{d}\mathbf{x},$$

and thus

$$\sum_{K \in \mathcal{T}} \int_K (\phi \, \Delta v - v \, \Delta \phi) \mathrm{d}\mathbf{x} = 0.$$

Since $v \in C^2(\overline{K})$, using Green's second identity (1.32) on each element implies that

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \left(\phi \frac{\partial v}{\partial n} - v \frac{\partial \phi}{\partial n} \right) \mathrm{d}s = 0.$$

Now, since $\phi \in C^\infty_{\text{comp}}(\Omega)$ and $v \in C(\overline{\Omega})$, $\sum_{K \in \mathcal{T}} \int_{\partial K} v \, \partial \phi \partial n \, \mathrm{d}s = 0$ (the integrals over interior edges cancel and ϕ is zero on $\partial\Omega$); thus we are left with

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \phi \frac{\partial v}{\partial n} \mathrm{d}s = 0.$$

Since ϕ is an arbitrary member of $C^\infty_{\text{comp}}(\Omega)$, this last equation can only hold if $\partial v / \partial n$ is continuous across each edge. Continuity of the tangential part of ∇v follows from the fact that v is continuous across edges, and thus $\nabla v \in C(\overline{\Omega})$. $\square$

For any conforming subspace, the continuity and coercivity properties of the new formulations imply that the corresponding Galerkin methods are quasi-optimal without any constraint on the subspace dimension, albeit with the constant in front of the best approximation error growing with k. (For simplicity we state this result for the case of the interior impedance problem of Definition 3.1, but a completely analogous result is valid in the case of the truncated sound-soft scattering problem of Definition 4.2.)

Proposition 5.2 (Quasi-optimality). *Suppose that the interior impedance problem of Definition 3.1 is solved using the variational formulation of Proposition 3.2 (with the constants α_j, β_j, and A chosen as in Lemma 3.3 and Theorem 3.4, and β_1 chosen proportional to L), with V_N a finite dimensional subspace of V. Then there exists a $C_{qo} > 0$ (depending only on $d, \vartheta_*, \vartheta^*$ and γ) such that*

$$\|u - u_N\|_V \le C_{qo}\big(kL + (kL)^{-1}\big) \inf_{v_N \in V_N} \|u - v_N\|_V, \tag{5.1}$$

for any $N > 0$ and for all $k > 0$.

Proof. This is a simple consequence of Céa's Lemma (1.11), the coercivity result of Theorem 3.4, and the bound on the continuity constant given in Lemma 3.3. $\square$

That the factor in front of the best approximation error grows with k is somehow expected because of the *pollution effect* [40, §4.6]. Indeed, if this factor were bounded independently of k then we would have proved that this method did not suffer from the pollution effect (in the sense

of [2, Definition 2.1]) for any choice of V_N. However, it is widely believed that no FEM converging in h (in $d \geq 2$) can be pollution-free, as was proved for a wide class of generalised FEMs in [2, Theorem 4.6].

Since we have established quasi-optimality, we only need to consider the approximation of the solution, i.e. for what subspaces does the best approximation error on the right-hand side of (5.1) tend to zero as $N \to \infty$.

Given a family $\mathbb{V} = \{V_N\}_{N \in \mathbb{N}}$ of finite dimensional subspaces of V, a norm $\|\cdot\|_W$ and $W := \{w \in V \text{ s.t. } \|w\|_W < \infty\}$, we say that $\mathbb{V}$ *approximates* W if

$$\lim_{N \to \infty} \inf_{v_N \in V_N \cap W} \|w - v_N\|_W = 0 \qquad \text{for all } w \in W.$$

Lemma 5.3. *A family of C^1-elements that approximates $H^2(\Omega)$ also approximates V.*

Proof. In Appendix A we prove that $\mathcal{D}(\overline{\Omega})$ is dense in V, so given $u \in V$ and $\varepsilon > 0$ there exists a $w \in \mathcal{D}(\overline{\Omega})$ such that $\|u - w\|_V < \varepsilon/2$. From the inclusion $H^2(\Omega) \subseteq V$ there exists a constant C such that $\|v\|_V \leq C \|v\|_{H^2(\Omega)}$ for every $v \in H^2(\Omega)$. From Lemma 5.1 and the approximation property in $H^2(\Omega)$, for N large enough we can choose a C^1-element function v_N (which also belongs to $H^2(\Omega)$) such that $\|w - v_N\|_{H^2(\Omega)} < \varepsilon/(2C)$. By the triangle inequality $\|u - v_N\|_V < \varepsilon$. $\qquad \square$

Conditions for polynomial C^1-elements to be dense in $H^2(\Omega)$ are given in [16, Theorem 48.2], and rates of convergence under the assumption of additional regularity are given in [16, Figure 48.1, Page 297] and [7, Table 3, § II.6]. Note that for the standard variational formulations of Laplace and Helmholtz problems one aims to prove convergence for solutions that belong to $H^{1+s}(\Omega)$ for some $0 < s \leq 1$ and then obtain a convergence rate for solutions in $H^2(\Omega)$ (see e.g. [8, Theorem 5.4.4]). For the new formulation, however, the standard results cited above only give a convergence rate for solutions at least in $H^3(\Omega)$.

An alternative to using piecewise-polynomial basis functions would be to use oscillatory basis functions from the Partition of Unity FEM [45], with the partition of unity chosen so that the elements are C^1-conforming. Convergence rates for plane or spherical wave bases can then be obtained by slightly modifying the proof of [45, Theorem 2.1] and using the approximation results in [44, §8.4], [49, Chapter 3].

5.2   Discrete conditioning and convergence of iterative solvers

Assume we have a conforming finite element method, with family of subspaces $V_N = \text{span}\{\phi_i : i = 1, \ldots, N\} \subset V$. (We also denote the subspaces V_h when we are explicitly thinking of them as coming from a mesh with meshwidth h.)

Let $b(\cdot, \cdot)$ and $G(\cdot)$ be the sesquilinear form and the antilinear functional introduced either in Section 3 or in Section 4. Define

$$\mathbf{B}_{ij} := b(\phi_j, \phi_i) \quad \text{and} \quad \mathbf{g}_i := G(\phi_i) \quad \text{for } i, j = 1, \ldots, N.$$

Then, if $u_N = \sum_{i=1}^N U_i \phi_i$, $v_N = \sum_{i=1}^N V_i \phi_i$, $\mathbf{u} = (U_i) \in \mathbb{C}^N$, and $\mathbf{v} = (V_i) \in \mathbb{C}^N$, the standard properties of sesquilinear forms imply that

$$(\mathbf{Bu}, \mathbf{v}) = b(u_N, v_N), \tag{5.2}$$

where $(\cdot, \cdot)$ denotes the standard Euclidean inner product for vectors in $\mathbb{C}^N$; thus the Galerkin method is equivalent to solving the linear system

$$\mathbf{Bu} = \mathbf{g}.$$

For simplicity we only consider the h-version of the FEM. We use the notation $a \lesssim b$ to mean $a \leq c\,b$ where c is independent of h, k, and L, and $a \sim b$ to mean that both $a \lesssim b$ and $b \lesssim a$.

Proposition 5.4 (Bounds on the discrete condition number). *Let $\mathcal{T}^h, 0 < h \leq 1$, be a quasi-uniform family of triangulations of Ω (in the sense of [16, Pages 61 and 135]) and let $V_h \subset V$ consist of piecewise polynomials of degree $\leq p$ for some fixed p that are in $C^1(\overline{\Omega})$ with basis functions*

scaled such that $\|v_h\|^2_{L^2(\Omega)} \sim h^d\|\mathbf{v}\|^2$ for all $v_h \in V_h$. Then, if $hk \lesssim 1$, the condition number $\kappa(\mathbf{B}) := \|\mathbf{B}\|_2 \, \|\mathbf{B}^{-1}\|_2$ satisfies

$$\kappa(\mathbf{B}) \lesssim \frac{1}{h^4 k^2} \left(L + \frac{1}{k}\right)\left(L + \frac{1}{k^2 L}\right). \tag{5.3}$$

Proof. If a sesquilinear form $b(\cdot, \cdot)$ is continuous and coercive with constants C_c and α respectively (as in (1.7) and (1.8)), and $M_1, M_2 > 0$ are such that

$$M_1\|\mathbf{v}\|_2^2 \le \|v_h\|_V^2 \le M_2\|\mathbf{v}\|_2^2 \qquad \text{for all } \mathbf{v} \in \mathbb{C}^N, \tag{5.4}$$

then the relationship (5.2) implies that, for all $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$,

$$|(\mathbf{B}\mathbf{u}, \mathbf{v})| \le M_2 C_c \|\mathbf{u}\|\|\mathbf{v}\| \quad \text{and} \quad |(\mathbf{B}\mathbf{v}, \mathbf{v})| \ge M_1 \alpha \|\mathbf{v}\|^2; \qquad \text{thus} \quad \kappa(\mathbf{B}) \le \frac{M_2 C_c}{M_1 \alpha}. \tag{5.5}$$

The bounds on the continuity constant and coercivity constant of $\mathbf{B}$ given by Lemma 3.3 and Theorem 3.4, respectively, imply that (as in the proof of Proposition 5.2)

$$\frac{C_c}{\alpha} \lesssim kL + \frac{1}{kL};$$

thus we only need to find M_1, M_2 such that the norm equivalence (5.4) holds.

Now

$$\|v_h\|_V^2 \ge k^2 \, \|v_h\|_{L^2(\Omega)}^2 \sim k^2 h^d \, \|\mathbf{v}\|^2,$$

so the first inequality in (5.4) holds for some $M_1 \gtrsim k^2 h^d$.

To obtain an upper bound on $\|v_h\|_V^2$ in terms of $\|\mathbf{v}\|^2$ we use the inverse estimates (e.g. [8, Lemma 4.5.3], [7, II.6.8], [16, Theorem 17.2])

$$|v_h|_{H^s(\Omega)} \lesssim h^{-s} \, \|v_h\|_{L^2(\Omega)} \quad \text{for all } v_h \in V_h, \quad s = 1, 2,$$

(note that this is where we need the assumptions that the mesh is quasi-uniform and that the basis is piecewise-polynomial), the multiplicative trace inequality ([33, Theorem 1.5.1.10, last formula on Page 41], [8, Theorem 1.6.6])

$$\|v\|_{L^2(\partial\Omega)}^2 \lesssim \|v\|_{L^2(\Omega)} \left(L^{-1} \, \|v\|_{L^2(\Omega)} + |v|_{H^1(\Omega)}\right) \quad \text{for all } v \in H^1(\Omega),$$

and the fact that $\|v_h\|_{L^2(\Omega)}^2 \sim h^d\|\mathbf{v}\|^2$. The result is that the second inequality in (5.4) holds for some M_2 such that

$$M_2 \lesssim \left(\frac{1}{h^2} + k^2 + \frac{1}{h^4 k^2} + k^2 + k^2\frac{L}{h} + \frac{1}{h^2} + \frac{L}{h^3}\right) h^d \lesssim \frac{1}{h^4 k}\left(L + \frac{1}{k}\right) h^d,$$

where in the second inequality we used $h \le L$ and $hk \lesssim 1$. Combining the bounds on C_c/α, M_1 and M_2 yields the result. $\qquad\square$

Remark 5.5 (Discussion of the bound on the condition number)**.** *There are two interesting limits under which to consider the bound (5.3): $h \to 0$ and $k \to \infty$. In the limit $h \to 0$ for fixed k, $\kappa(\mathbf{B}) \sim h^{-4}$; this is expected because of the presence of Δv in the norm and the consequent use of inverse estimates for the H^2-seminorm (cf. FEMs for the biharmonic problem). In the limit $k \to \infty$, we need to tie h to k, since if h is fixed then the best approximation error is not bounded as $k \to \infty$. It is commonly believed that $hk \sim 1$ keeps the relative best approximation error bounded as k increases, although this has only been rigorously proved for certain 1-d problems [40, Equation 4.4.3] and [12, Lemma 6.6]. Under the scaling $hk \sim 1$, $\kappa(\mathbf{B}) \sim k^2$ as $k \to \infty$ (although from Proposition 5.2 we expect some pollution in this limit, and thus some loss of accuracy of the Galerkin solution at large k). There do not yet exist any comparable results about the conditioning of the standard formulation (1.19) to compare this to.*

As discussed in §1, the sign-indefiniteness of the standard variational formulations of the Helmholtz equation is a major issue when solving the resulting linear systems with iterative solvers such as GMRES. We now briefly investigate whether or not we can determine anything a priori about how GMRES behaves when applied to $\mathbf{Bu} = \mathbf{g}$. Of course, linear systems arising from FEMs are usually preconditioned before being solved using GMRES (for a description of the state-of-the-art preconditioners for the Helmholtz equation with large k see [26], [27], [25], [1]), however here we just consider applying GMRES to the unpreconditioned problem.

We use the fact that, for $m \in \mathbb{N}$, the GMRES residual $\mathbf{r}_m$ satisfy

$$\frac{\|\mathbf{r}_m\|}{\|\mathbf{r}_0\|} \leq \sin^m \beta, \quad \text{where} \quad \cos \beta = \frac{\text{dist}(0, W(\mathbf{B}))}{\|\mathbf{B}\|}, \tag{5.6}$$

where $W(\mathbf{B}) := \{(\mathbf{Bx}, \mathbf{x}) : \mathbf{x} \in \mathbb{C}^N, \|\mathbf{x}\| = 1\}$ is the *numerical range* of $\mathbf{B}$. This bound was originally proved in [23] (see also [22, Theorem 3.3]) and appears in the form above in [3, Equation (1.2)].

It follows from (5.5) that $\cos \beta \geq M_1 \alpha / (M_2 C_c)$ where M_1 and M_2 are as in (5.4). Using the bounds on M_1 and M_2 in the proof of Proposition 5.4, one can then prove that, given $\varepsilon > 0$ and $k_0 > 0$, there exists a $C_1 > 0$ independent of k, L, and ε such that

$$m \geq \frac{C_1 L^4}{h^8 k^4} |\log \varepsilon| \quad \text{implies that} \quad \frac{\|\mathbf{r}_m\|}{\|\mathbf{r}_0\|} \leq \varepsilon, \tag{5.7}$$

for all $k \geq k_0$. To understand this bound better, consider the case $hk \sim 1$ (which, as discussed above is thought to keep the relative best approximation error under control) and ignore the dependence on L; the bound then becomes $m \gtrsim k^4$. Unfortunately this bound is not practical, since if $hk \sim 1$ then $N \sim k^d$, and (in exact arithmetic) GMRES always converges once the number of iterations, m, reaches the dimension N of the linear system. It is instructive to note that two of the powers of k in $m \gtrsim k^4$ arise from the fact that $C_c / \alpha \sim k$, and two powers come from the norm in V, so even if the method were pollution-free, i.e. if C_c / α were bounded independently of k, then the estimate (5.6) would give $m \gtrsim k^2$, which is still not particularly useful. (Similarly, a hypothetical H^1-conforming scheme with continuity and coercivity properties similar to those of Section 3 would also give $m \gtrsim k^2$.)

In summary, although the bound (5.6) allows us to determine k-explicit a priori information about the behaviour of (unpreconditioned) GMRES from the continuity and coercivity properties of the new formulation, the resulting bounds do not yield any practical information when k is large.

6 Discussion of the geometric restrictions on the new formulations

The new formulations in Sections 3 and 4 both require that certain domains be star-shaped with respect to a ball. In this section we discuss whether these restrictions can be lifted. This is perhaps more easily understandable for the exterior problem, so we begin here.

6.1 The sound-soft scattering problem

The coercive formulation of the truncated sound-soft scattering problem in Section 4 needed both Ω_D (the obstacle) and Ω_R (the interior of the artificial boundary) to be star-shaped with respect to a ball. Indeed, the proof of coercivity required that $\mathbf{x} \cdot \mathbf{n}_D(\mathbf{x}) > 0$ for $\mathbf{x} \in \Gamma_D$ and similarly for $\mathbf{x} \in \Gamma_R$. Replacing the vector field $\mathbf{x}$ in the identity (1.27) by an arbitrary vector field $\mathbf{Z}(\mathbf{x})$, one can show that there exists a coercive formulation of the truncated sound-soft scattering problem, for k sufficiently large, if there exists a $\mathbf{Z}(\mathbf{x})$ such that

$\mathbf{Z}(\mathbf{x}) \cdot \mathbf{n}_D(\mathbf{x}) > 0$ for $\mathbf{x} \in \Gamma_D$, $\hfill$ (6.1a)

$\mathbf{Z}(\mathbf{x}) \cdot \mathbf{n}_R(\mathbf{x}) > 0$ for $\mathbf{x} \in \Gamma_R$, $\hfill$ (6.1b)

there exists a $\theta > 0$ such that $\Re\{\partial_i \mathbf{Z}_j(\mathbf{x}) \xi_i \overline{\xi_j}\} \geq \theta |\xi|^2$ for all $\xi \in \mathbb{C}^d$ and for all $\mathbf{x} \in \Omega$, $\hfill$ (6.1c)

where the last condition ensures strict positivity of the volume terms of the sesquilinear form. The choice $\mathbf{Z}(\mathbf{x}) = \mathbf{x}$ satisfies these conditions when Ω_D and Ω_R are star-shaped with respect to a ball; for what other domains does such a $\mathbf{Z}(\mathbf{x})$ exist? Note that since the choice of Ω_R is up to us when using the truncated problem to approximate the full scattering problem, we are really interested in obtaining an appropriate $\mathbf{Z}(\mathbf{x})$ for a wider class of Ω_D.

For Helmholtz problems in domains exterior to a bounded obstacle, the key geometric condition is that of *non-trapping*. Roughly speaking, an exterior domain is non-trapping if any ray hitting the obstacle and then reflecting with the angle of incidence equal to the angle of reflection eventually escapes to infinity (after multiple reflections if necessary). For example, one can show that star-shaped domains are non-trapping. In contrast, trapping domains can "trap" certain rays in a neighbourhood of the obstacle for an arbitrary long time (the review [12, §5.2] contains a more precise discussion of trapping and non-trapping, aimed at numerical analysts but with references to the more technical definitions).

Morawetz, Ralston, and Strauss [55, §4] showed that if $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_D}$ is a 2-dimensional non-trapping domain, then, with Ω_R the ball of radius R for some sufficiently large $R > 0$, there exists a $\mathbf{Z}(\mathbf{x})$ in $\Omega := \Omega_R \setminus \overline{\Omega_D}$ such that

$$\mathbf{Z}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})_D > 0 \text{ for } \mathbf{x} \in \Gamma_D, \tag{6.2a}$$

$$\mathbf{Z}(\mathbf{x}) = \mathbf{x} \text{ for } \mathbf{x} \in \Gamma_R, \tag{6.2b}$$

$$\Re\{\partial_i \mathbf{Z}_j(\mathbf{x})\xi_i\overline{\xi_j}\} \geq 0 \text{ for all } \xi \in \mathbb{C}^d \text{ and for all } \mathbf{x} \in \Omega. \tag{6.2c}$$

This $\mathbf{Z}(\mathbf{x})$ satisfies (6.1a) and (6.1b), but not quite (6.1c). Although it is not immediately clear whether the construction of the $\mathbf{Z}(\mathbf{x})$ of [55, §4] can be suitably modified to obtain a $\mathbf{Z}(\mathbf{x})$ satisfying the more stringent requirements (6.1), the similarity of the conditions (6.1) and (6.2) indicates that it is reasonable to believe that a coercive formulation of the truncated sound-soft scattering problem exists if Ω_+ is non-trapping (or perhaps satisfies a slightly more restrictive condition). However, although the existence of a $\mathbf{Z}(\mathbf{x})$ satisfying (6.2) is shown constructively in [55, §4], it is not immediately clear how easily it can be evaluated numerically (which would be a requirement if a variational formulation involving a similar $\mathbf{Z}(\mathbf{x})$ were to be implemented practically).

In addition, there is a good reason to believe that coercive formulations *cannot* exist for trapping domains (or at least not formulations that are coercive uniformly in k). Indeed, one of the consequences of coercivity is the bound on the solution (4.9). For the sound-soft scattering problem, Definition 4.1, in non-trapping domains an analogous bound holds, with the norm of the solution (weighted with k as in (1.22)) bounded uniformly by norms of the data [64], [53] (see the discussion in [12, Theorem 5.6 and Remark 5.9]). However, for certain trapping domains the norm of the solution operator can grow exponentially with k (see, e.g., [12, §5.6.2, Page 221]). Thus, if a coercive formulation of the truncated sound soft scattering problem existed for these trapping domains, and $b(\cdot,\cdot)$ and $G(\cdot)$ were normalised so that $\|G\|_{V'} \lesssim \|f\|_{L^2(\Omega)} + \|g_R\|_{L^2(\Gamma_R)} + k\|g_D\|_{L^2(\Gamma_D)} + \|\nabla_{\Gamma_D} g_D\|_{L^2(\Gamma_D)}$, with the omitted constant independent of k (as in the formulation of §4), then the coercivity constant would have to decrease exponentially with k.

6.2   The interior impedance problem

The coercive formulation of the interior impedance problem in Section 3 required the bounded domain Ω to be star-shaped with respect to a ball, with the fact that $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) > 0$ used often in the proof of coercivity. Similar to above, replacing the vector field $\mathbf{x}$ in the identity (1.27) by $\mathbf{Z}(\mathbf{x})$, one can show that there exists a coercive formulation of the interior impedance problem if there exists a $\mathbf{Z}(\mathbf{x})$ such that

$$\mathbf{Z}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0 \text{ for } \mathbf{x} \in \Gamma, \tag{6.3a}$$

there exists a $\theta > 0$ such that $\Re\{\partial_i \mathbf{Z}_j(\mathbf{x})\xi_i\overline{\xi_j}\} \geq \theta|\xi|^2$ for all $\xi \in \mathbb{C}^d$ and for all $\mathbf{x} \in \Omega$. (6.3b)

The choice $\mathbf{Z}(\mathbf{x}) = \mathbf{x}$ satisfies these conditions for Ω that are star-shaped with respect to a ball. It is not clear, however, how to construct such a $\mathbf{Z}$ for more general domains; although it is straightforward to construct a $\mathbf{Z}$ satisfying (6.3a) (see [33, Lemma 1.5.1.9], [65, Theorem 1.12

(vi)]), satisfying (6.3b) is much more difficult. (Note that the impedance boundary condition corresponds to the boundary taking energy away from any impinging wave, and thus the concepts of trapping and non-trapping, relying on energy conservation, do not apply to this problem.)

Regarding bounds on the solution in terms of the data, the currently best available ones for the interior impedance problem in general Lipschitz domains have positive powers of k on the right-hand sides [28, Theorem 2.4]. Whether these bounds are sharp is not yet known; if they are sharp, then any formulation that is coercive for general Lipschitz domains would have its coercivity constant decreasing at least polynomially in k (assuming $b(\cdot, \cdot)$ and $G(\cdot)$ are normalised such that $\|G\|_{V'} \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}$ with the omitted constant independent of k).

7 Concluding remarks

This paper began by questioning whether the Helmholtz equation should be described as "sign-indefinite". The fact remains that the standard variational formulations of the Helmholtz equation are sign-indefinite. However, we hope that by introducing the sign-definite formulations in this paper, which are obtained in a manner similar to how the standard variational formulations are obtained (i.e. by multiplying by a test function and integrating by parts), we will at least make the reader hesitate if they ever find themselves writing "the Helmholtz equation is sign-indefinite"!

Acknowledgements

A   Appendix: Density of $\mathcal{D}(\overline{\Omega})$ in the space V

Lemma A.1. *Let Ω be a bounded Lipschitz domain. Then $\mathcal{D}(\overline{\Omega}) := \{U|_\Omega : U \in C^\infty(\mathbb{R}^d)\}$ is dense in the space V defined by (1.21).*

Proof. Note to begin with that in this proof we use γ to denote the trace operator $H^s(\Omega) \to H^{s-1}(\partial\Omega)$ for $1/2 < s < 3/2$ (see, e.g, [43, Theorem 3.38, Page 102]). We also use the notation $\mathcal{D}(\Omega)$ for $C^\infty_{\text{comp}}(\Omega) = \{v \in C^\infty(\Omega), \text{supp}\, v \subset\subset \Omega\}$.

Via a partition of unity it is sufficient to consider the case of a Lipschitz hypograph, i.e.

$$\Omega := \big\{(x', x_d) \subset \mathbb{R}^d : x' \in \mathbb{R}^{d-1}, x_d > f(x')\big\},$$

where $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is in $C^{0,1}(\mathbb{R}^{d-1})$ (for examples of this method of arguing, see, e.g. [12, §A.2] and [43, Page 89 onwards]). Since Ω is now unbounded, we define V as the space of functions u such that $\|u\psi\|_V < \infty$, for any $\psi \in \mathcal{D}(\overline{\Omega})$, where $\|\cdot\|_V$ is defined by (1.22).

The main idea of the proof is that a given $u \in V$ can be approximated by u_t where, for $t > 0$,

$$u_t(\mathbf{x}) := u(\mathbf{x} + t\mathbf{e}_d),$$

where $\mathbf{e}_d$ is the unit vector in the direction x_d. Thus, for $\mathbf{x} \in \partial\Omega$, $u_t(\mathbf{x})$ is u evaluated on a parallel surface to $\partial\Omega$, at a distance t above. Now, by standard interior regularity results for the Laplacian applied to bounded subsets of Ω (see, e.g., [29, §6.3.1, Page 309], [43, Theorem 4.16, Page 135]), we have that $u \in H^2_{\text{loc}}(\Omega)$, i.e. $\chi u \in H^2(\Omega)$ for every $\chi \in \mathcal{D}(\Omega)$, and thus $\psi u_t \in H^2(\Omega)$ for every $\psi \in \mathcal{D}(\overline{\Omega})$. The key point is that u_t and all its derivatives of order ≤ 2 are square-integrable in any

bounded subset of Ω (including those that share part of their boundary with Ω) but this is not in general true for u.

The main part of the proof consists of showing below that $\|(u - u_t)\psi\|_V \to 0$ as $t \to 0$, for any $\psi \in \mathcal{D}(\overline{\Omega})$. Assuming this result holds, given $\varepsilon > 0$ and $\psi \in \mathcal{D}(\overline{\Omega})$, there exists a $t > 0$ such that $\|(u - u_t)\psi\|_V < \varepsilon/2$. Let C_ψ be such that $\|v\psi\|_V \leq C_\psi \|v\psi\|_{H^2(\mathrm{supp}\,\psi)}$ for every $v \in H^2(\mathrm{supp}\,\psi)$. Since the restriction of $\mathcal{D}(\overline{\Omega})$ is dense in $H^2(\mathrm{supp}\,\psi)$ [43, Page 77], there exists a $w \in \mathcal{D}(\overline{\Omega})$ such that $\|(w - u_t)\psi\|_{H^2(\Omega)} < \varepsilon/(2C_\psi)$. Noting that $\|(w - u_t)\psi\|_V \leq C_\psi\|(w - u_t)\psi\|_{H^2(\Omega)} < \varepsilon/2$, we then have that $\|(u - w)\psi\|_V < \varepsilon$ by the triangle inequality, and so we are done.

Thus, we need only prove that, for all $\psi \in \mathcal{D}(\overline{\Omega})$, $\|(u - u_t)\psi\|_V \to 0$ as $t \to 0$. We now show that, for an arbitrary $\psi \in \mathcal{D}(\overline{\Omega})$, each of the terms in $\|(u - u_t)\psi\|_V$ tends to zero as $t \to 0$.

We first show that $\|(u - u_t)\psi\|_{H^1(\Omega)} \to 0$ as $t \to 0$. To do this, choose $v \in H^1(\mathbb{R}^d)$ such that $v|_{\Omega_1} = u$, where $\Omega_1 := \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x}, \mathrm{supp}\,\psi) < 1\}$. Then define $v_t(\mathbf{x}) := v(\mathbf{x} + t\mathbf{e}_d)$ for $t > 0$, and thus $v_t|_{\mathrm{supp}\,\psi} = u_t$ for any $0 < t < 1$. These definitions immediately imply that, for $0 < t < 1$ and some $C > 0$,

$$\|(u - u_t)\psi\|_{H^1(\Omega)} \leq C\,\|u - u_t\|_{H^1(\mathrm{supp}\,\psi)} \leq C\,\|v - v_t\|_{H^1(\mathbb{R}^d)}. \tag{A.1}$$

If $v \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ then $\|v - v_t\|_{H^1(\mathbb{R}^d)} \to 0$ as $t \to 0$, and thus, by the density of $C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$, this is also true for $v \in H^1(\mathbb{R}^d)$. The inequality (A.1) then implies that $\|(u-u_t)\psi\|_{H^1(\Omega)} \to 0$ as $t \to 0$.

In order to show that $\|\Delta((u - u_t)\psi)\|_{L^2(\Omega)} \to 0$ as $t \to 0$, we only need to show that $\|(\Delta u - \Delta u_t)\psi\|_{L^2(\Omega)} \to 0$, since the terms involving $(u - u_t)\Delta\psi$ and $\nabla(u - u_t) \cdot \nabla\psi$ are bounded by $\|u - u_t\|_{H^1(\mathrm{supp}\,\psi)}$, which tends to zero by the previous paragraph. Thus, we need to show that

$$\int_\Omega (\Delta u - \Delta u_t)\psi\phi\,\mathrm{d}\mathbf{x} \to 0 \quad \text{for all } \phi \in L^2(\Omega),$$

and since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ we only need prove this for $\phi \in \mathcal{D}(\Omega)$. By the definition of the weak derivative and the Cauchy–Schwarz inequality, for $\phi \in \mathcal{D}(\Omega)$,

$$\left|\int_\Omega (\Delta u - \Delta u_t)\psi\phi\,\mathrm{d}\mathbf{x}\right| = \left|\int_\Omega (u - u_t)\Delta(\psi\phi)\,\mathrm{d}\mathbf{x}\right|$$
$$= \left|\int_{\mathrm{supp}\,\psi} (u - u_t)\Delta(\psi\phi)\,\mathrm{d}\mathbf{x}\right| \leq \|u - u_t\|_{L^2(\mathrm{supp}\,\psi)}\|\Delta(\psi\phi)\|_{L^2(\mathrm{supp}\,\psi)},$$

which tends to zero as $t \to 0$.

Moving to the terms on the boundary, the L^2-trace of $(u - u_t)\psi$ converges by the continuity of the trace operator:

$$\left\|\gamma\big((u - u_t)\psi\big)\right\|_{L^2(\partial\Omega)} \leq \left\|\gamma\big((u - u_t)\psi\big)\right\|_{H^{1/2}(\partial\Omega)} \leq \|(u - u_t)\psi\|_{H^1(\Omega)} \to 0. \tag{A.2}$$

To show that $\|\nabla_{\partial\Omega}\gamma((u-u_t)\psi)\|_{L^2(\partial\Omega)} \to 0$ we only need to show that $\|\psi\nabla_{\partial\Omega}(\gamma(u-u_t))\|_{L^2(\partial\Omega)} \to 0$ since the $\gamma(u-u_t)\nabla_{\partial\Omega}\psi$ term can be controlled using the mapping properties of the trace operator in a manner similar to that in (A.2).

In order to prove that $\|\psi\nabla_{\partial\Omega}(\gamma(u - u_t))\|_{L^2(\partial\Omega)} \to 0$, we only need to show that

$$\int_{\partial\Omega} \nabla_{\partial\Omega}(\gamma(u - u_t)) \cdot \psi\boldsymbol{\phi}\,\mathrm{d}s \to 0 \quad \text{as } t \to 0, \quad \text{for all } \boldsymbol{\phi} \in L^2_t(\partial\Omega),$$

where $L^2_t(\partial\Omega) := \{\boldsymbol{\phi} \in (L^2(\partial\Omega))^d : \mathbf{n} \cdot \boldsymbol{\phi} = 0\}$. Let $\nabla^*_{\partial\Omega} : L^2_t(\partial\Omega) \to (H^1(\partial\Omega))^*$ denote the adjoint of $\nabla_{\partial\Omega} : H^1(\partial\Omega) \to L^2_t(\partial\Omega)$. There exists a dense subspace X_t of $L^2_t(\partial\Omega)$ such that $\nabla^*_{\partial\Omega}(X_t) \subset L^2(\partial\Omega)$ (for explicit constructions of X_t in 2- and 3-d see [12, §A.3, Page 278]). Using this fact, and noting that the range of integration can be changed to $\mathrm{supp}\,\psi \cap \partial\Omega$, we only need to show that

$$\int_{\mathrm{supp}\,\psi \cap \partial\Omega} \gamma(u - u_t) \cdot \nabla^*_{\partial\Omega}(\psi\boldsymbol{\phi})\,\mathrm{d}s \to 0 \quad \text{as } t \to 0, \quad \text{for all } \boldsymbol{\phi} \in X_t.$$

However, this integral is bounded by $\|\gamma(u-u_t)\|_{L^2(\mathrm{supp}\,\psi\cap\partial\Omega)}\|\nabla^*_{\partial\Omega}(\psi\phi)\|_{L^2(\mathrm{supp}\,\psi\cap\partial\Omega)}$, which tends to zero as $t \to 0$ using arguments identical to those used in (A.2).

The last term to control is $\|(\partial((u-u_t)\psi)/\partial n\|_{L^2(\partial\Omega)}$. The regularity result of Nečas [56, §5.1.2], [43, Theorem 4.24 (i)] implies that this term can be bounded by a sum of all the previous terms. Indeed, this result (with the differential operator equal to the Laplacian) applied to the function $(u - u_t)\psi$ on the domain $\Omega' := \Omega \cap B_R$, with $R > 0$ chosen such that $\mathrm{supp}\,\psi \subset B_R$, implies that, for some $C > 0$,

$$\left\|\frac{\partial}{\partial n}\big((u-u_t)\psi\big)\right\|_{L^2(\partial\Omega)} = \left\|\frac{\partial}{\partial n}\big((u-u_t)\psi\big)\right\|_{L^2(\partial\Omega')}$$
$$\leq C\Big(\left\|(u-u_t)\psi\right\|_{H^1(\Omega')} + \left\|\Delta\big((u-u_t)\psi\big)\right\|_{L^2(\Omega')} + \left\|\gamma\big((u-u_t)\psi\big)\right\|_{H^1(\partial\Omega')}\Big),$$

which tends to zero as $t \to 0$; thus the proof is complete. $\qquad\square$

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