AN APPROXIMATE LINEARISED RIEMANN SOLVER FOR THE EULER EQUATIONS IN ONE DIMENSION WITH A GENERAL EQUATION OF STATE

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ABSTRACT

An approximate (linearised) Riemann solver is presented for the solution of the Euler equations of gas dynamics in one dimension with a general equation of state. The scheme is applied to a standard shock reflection test problem for some specimen equations of state.

1. INTRODUCTION

The linearised approximate Riemann solver of Roe [1] was proposed in 1981 for the solution of the Euler equations of gas dynamics where the properties of the fluid are represented by the ideal equation of state. We seek here to extend this scheme to the solution of the Euler equations in one dimension for a general equation of state. At each stage we shall as far as possible draw a parallel with Roe's scheme for the ideal equation of state. Results for the extended scheme are presented for a particular problem of shock reflection for three different equations of state.

In §2 we look at the Jacobian matrix of the flux function for the Euler equations with a general equation of state, and in §3 derive an approximate Riemann solver for the solution of these equations. In §4 we give some particular examples of non-ideal equations of state, and in §5 we describe a standard test problem involving shock reflection together with a derivation of the exact solution. Finally, in §6 we display the numerical results achieved for this test problem with three different equations of state.

A similar calculation with the same objective has been presented by Roe [2] which, however, differs in both procedure and final form.

2. EQUATIONS OF FLOW AND STATE

In this section we state the equations of motion for an inviscid compressible fluid in one dimension for any equation of state, and derive the eigenvalues and eigenvectors of the Jacobian of the corresponding flux function.

2.1 Equations

The Euler equations governing the flow of an inviscid, compressible fluid in one dimension can be written in conservation form as

$$\rho_t + (\rho u)_x = 0 \tag{2.1}$$

$$(\rho u)_t + (p + \rho u^2)_x = 0 \tag{2.2}$$

$$e_t + \{u(e + p)\}_x = 0 \quad , \tag{2.3}$$

together with

$$e = \rho i + \tfrac{1}{2}\rho u^2 \tag{2.4}$$

where $\rho = \rho(x,t)$, $u = u(x,t)$, $p = p(x,t)$, $i = i(x,t)$ and $e = e(x,t)$ represent the density, velocity, pressure, specific internal energy and the total energy, respectively, at a position x and time t . Equations (2.1)-(2.3) represent conservation of mass, momentum and energy, respectively. In addition, there is an equation of state which is a macroscopic, thermodynamic relationship specific to each particular fluid.

For most fluids we can assume a relation of the form

$$g(p,V,T) = 0 \tag{2.5}$$

where $V = 1/\rho$ is the specific volume and T is the absolute temperature. However, since there is a direct relationship between specific internal energy i and absolute temperature T, we rewrite equation (2.5) as

$$p = p(\rho,i) \quad . \tag{2.6}$$

Moreover, we assume that the first derivatives $\left.\frac{\partial p}{\partial \rho}\right|_i$ and $\left.\frac{\partial p}{\partial i}\right|_\rho$ are available. In the case of an ideal gas, equation (2.6) becomes

$$p = (\gamma - 1)\rho i \quad , \tag{2.7}$$

where γ is the ratio of specific heat capacities of the fluid: this is sometimes called a γ-gas law. The relationship given in equation (2.6) will usually be determined by experimental considerations.

In summary, we are interested in the system of hyperbolic equations

$$\underline{w}_t + \underline{F}_x = \underline{0} \quad , \tag{2.8}$$

where

$$\underline{w} = (\rho,\rho u,e)^T \tag{2.9a}$$

$$\underline{F}(\underline{w}) = (\rho u,p+\rho u^2,u(e+p))^T \tag{2.9b}$$

$$e = \rho i + \tfrac{1}{2}\rho u^2 \tag{2.9c}$$

with

$$p = p(\rho,i) \quad , \tag{2.9d}$$

where the particular form for equation (2.9d) is given.

2.2 Jacobian

We now construct the Jacobian, A, of the flux function, $F(\underline{w})$, given by

$$A = \frac{\partial \underline{F}}{\partial \underline{w}} \quad , \tag{2.10}$$

and find its eigenvalues and (right) eigenvectors since this will form the basis for our approximate Riemann solver.

Defining the momentum m as $m = \rho u$ we may rewrite the equations in the form

$$\underline{w} = (\rho, m, e)^T \tag{2.11a}$$

$$\underline{F}(\underline{w}) = \left(m, p + \frac{m^2}{\rho} \ , \ \frac{me}{\rho} + \frac{mp}{\rho}\right) \tag{2.11b}$$

and

$$p = p(\rho, i) \quad , \tag{2.11c}$$

where

$$i = \frac{e}{\rho} - \frac{1}{2}\frac{m^2}{\rho^2} \quad . \tag{2.11d}$$

Now,

$$\frac{\partial \underline{F}}{\partial \underline{w}} = \left(\left.\frac{\partial \underline{F}}{\partial \rho}\right|_{m,e} , \left.\frac{\partial \underline{F}}{\partial m}\right|_{\rho,e} , \left.\frac{\partial \underline{F}}{\partial e}\right|_{\rho,m}\right) \tag{2.12}$$

in particular, we will need to find

$$\left.\frac{\partial p}{\partial \rho}(\rho, i(\rho,m,e))\right|_{m,e} \quad , \quad \left.\frac{\partial p}{\partial m}(\rho, i(\rho,m,e))\right|_{\rho,e}$$

and $\left.\frac{\partial p}{\partial e}(\rho, i(\rho,m,e))\right|_{\rho,m}$. By the chain rule for partial derivatives, however, we have

$$\left.\frac{\partial p}{\partial \rho}(\rho, i(\rho,m,e))\right|_{m,e} = \left.\frac{\partial p}{\partial \rho}(\rho,i)\right|_{i} + \left.\frac{\partial i}{\partial \rho}(\rho,m,e)\right|_{m,e} \left.\frac{\partial p}{\partial i}(\rho,i)\right|_{\rho} \tag{2.13a}$$

$$\frac{\partial p}{\partial m}(\rho, i(\rho,m,e))\Big|_{\rho,e} = \frac{\partial i}{\partial m}(\rho,m,e)\Big|_{\rho,e} \; \frac{\partial p}{\partial i}(\rho,i)\Big|_{\rho} \tag{2.13b}$$

$$\frac{\partial p}{\partial e}(\rho, i(\rho,m,e))\Big|_{\rho,m} = \frac{\partial i}{\partial e}(\rho,m,e)\Big|_{\rho,m} \; \frac{\partial p}{\partial i}(\rho,i)\Big|_{\rho} \tag{2.13c}$$

where

$$i = i(\rho,m,e) = \frac{e}{\rho} - \frac{1}{2}\frac{m^2}{\rho^2} \; . \tag{2.14}$$

This leads to the following expression for the Jacobian

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a^2 - u^2 - \dfrac{p_i}{\rho}(H - u^2) & 2u - \dfrac{up_i}{\rho} & \dfrac{p_i}{\rho} \\ u(a^2 - H) - \dfrac{up_i}{\rho}(H - u^2) & H - \dfrac{u^2 p_i}{\rho} & u + \dfrac{up_i}{\rho} \end{bmatrix} \tag{2.15}$$

where the entralpy, H , is defined by

$$H = \frac{e+p}{\rho} = \frac{p}{\rho} + i + \frac{1}{2}u^2 \quad , \tag{2.16}$$

the 'sound speed', a , is given by

$$a = \frac{pp_i}{\rho^2} + p_\rho \tag{2.17}$$

and the shorthand notation $p_\rho \equiv \frac{\partial p}{\partial \rho}(\rho,i)\Big|_i$, $p_i \equiv \frac{\partial p}{\partial i}(\rho,i)\Big|_\rho$ has been used.

The eigenvalues, λ_i , and corresponding right eigenvectors, $\underline{e}_i$, of A are then found to be

$$\lambda_1 = u + a \;, \quad \underline{e}_1 = \begin{pmatrix} 1 \\ u + a \\ H + ua \end{pmatrix} = \begin{pmatrix} 1 \\ u + a \\ \frac{p}{\rho} + i + \frac{1}{2} u^2 + ua \end{pmatrix} \;, \tag{2.18a}$$

$$\lambda_2 = u - a \;, \quad \underline{e}_2 = \begin{pmatrix} 1 \\ u - a \\ H + ua \end{pmatrix} = \begin{pmatrix} 1 \\ u - a \\ \frac{p}{\rho} + i + \frac{1}{2} u^2 - ua \end{pmatrix} \;, \tag{2.18b}$$

and

$$\lambda_3 = u \;, \qquad \underline{e}_3 = \begin{pmatrix} 1 \\ u \\ H - \frac{\rho a^2}{p_i} \end{pmatrix} = \begin{pmatrix} 1 \\ u \\ i + \frac{1}{2} u^2 - \frac{\rho p_\rho}{p_i} \end{pmatrix} \;. \tag{2.18c}$$

We note that in the case of an ideal gas the equation of state (2.11c) becomes

$$p = (\gamma - 1)\rho i \tag{2.19}$$

giving

$$p_i = (\gamma - 1)\rho \;, \quad p_\rho = (\gamma - 1)\, i \tag{2.20}$$

and thus

$$\frac{a^2}{\gamma-1} = \frac{p}{\rho} + i = H - \frac{1}{2} u^2 \;. \tag{2.21}$$

In particular, the eigenvector $\underline{e}_3$ becomes

$$\underline{e}_3 = \begin{pmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{pmatrix} \;. \tag{2.22}$$

In the next section we develop an approximate Riemann solver using the results in this section.

3. AN APPROXIMATE RIEMANN SOLVER

In this section we develop an approximate Riemann solver for the Euler equations in one dimension with a general equation of state. We follow a similar course of reasoning as that used by Roe [3].

We consider two states $\underline{w}_L$, $\underline{w}_R$ (left and right) close to an average state $\underline{w}$, and seek α_1, α_2, α_3, such that

$$\Delta\underline{w} = \sum_{j=1}^{3} \alpha_j \underline{e}_j \tag{3.1}$$

to within $O(\Delta^2)$, where $\Delta(\circ) = (\circ)_R - (\circ)_L$. Writing equation (3.1) out in full we have

$$\Delta\rho = \alpha_1 + \alpha_2 + \alpha_3 \tag{3.2a}$$

$$\Delta(\rho u) = \alpha_1(u+a) + \alpha_2(u-a) + \alpha_3 u \tag{3.2b}$$

$$\Delta e = \alpha_1\left(\frac{p}{\rho} + i + \tfrac{1}{2}u^2 + ua\right) + \alpha_2\left(\frac{p}{\rho} + i + \tfrac{1}{2}u^2 - ua\right) + \alpha_3\left(i + \tfrac{1}{2}u^2 - \frac{\rho p_\rho}{p_i}\right) . \tag{3.2c}$$

From equations (3.2a-b) we have that

$$\Delta(\rho u) - u\Delta\rho = a(\alpha_1 - \alpha_2) \tag{3.3}$$

and from equations (3.2a), (3.2c)

$$\Delta(\rho i) - i\Delta\rho + \Delta\left(\frac{\rho u^2}{2}\right) - \tfrac{1}{2}u^2\Delta\rho = \frac{p}{\rho}(\alpha_1+\alpha_2) + ua(\alpha_1-\alpha_2) - \alpha_3\frac{\rho p_e}{p_i} . \tag{3.4}$$

Using equation (3.3) together with $\alpha_1 + \alpha_2 = \Delta\rho - \alpha_3$ equation (3.4) yields the following equation for α_3 :

$$\frac{\alpha_3}{p_i}\left(\frac{pp_i}{\rho} + \rho p_e\right) = i\Delta\rho - \Delta(\rho i) + \frac{p}{\rho}\Delta\rho - \frac{u^2}{2}\Delta\rho - \Delta\left(\frac{\rho u^2}{2}\right) + u\Delta(\rho u) , \tag{3.5}$$

and since

$$\rho a^2 = \frac{pp_i}{\rho} + \rho p_\rho \tag{3.6}$$

α_3 is given by

$$\rho a^2 \frac{\alpha_3}{p_i} = i\Delta\rho - \Delta(\rho i) + \frac{p}{\rho}\Delta\rho - \frac{u^2}{2}\Delta\rho - \Delta\left(\frac{\rho u^2}{2}\right) + u\Delta(\rho u) . \tag{3.7a}$$

In addition, α_1 and α_2 can now be calculated from equations (3.2a) and (3.3), i.e.

$$\alpha_1 + \alpha_2 = \Delta\rho - \alpha_3 \tag{3.7b}$$

$$\alpha_1 - \alpha_2 = \frac{\Delta(\rho u) - \Delta\rho}{a} . \tag{3.7c}$$

We have made the assumption that the left and right states $\underline{w}_L$, $\underline{w}_R$ are close to some average state $\underline{w}$ to within $O(\Delta^2)$, so that, to this approximation

$$\Delta(\rho u) = u\Delta\rho + \rho\Delta u \tag{3.8a}$$

$$\Delta(\rho i) = i\Delta\rho + \rho\Delta i \tag{3.8b}$$

$$\Delta(\rho u^2) = u^2\Delta\rho + 2\rho u\Delta u . \tag{3.8c}$$

In that case equation (3.7a) gives

$$\rho a^2 \frac{\alpha_3}{p_i} = \frac{p}{\rho}\Delta\rho - \rho\Delta i , \tag{3.9}$$

and using equation (3.6) we obtain

$$\alpha_3 = \Delta\rho - \frac{(p_\rho \Delta\rho + p_i \Delta i)}{a^2} \quad . \tag{3.10}$$

But

$$\Delta p = p_\rho \Delta\rho + p_i \Delta i \tag{3.11}$$

to within $O(\Delta^2)$, and therefore

$$\alpha_3 = \Delta\rho - \frac{\Delta p}{a^2} \quad . \tag{3.12}$$

Finally, equations (3.7b-c) now become

$$\alpha_1 + \alpha_2 = \frac{\Delta p}{a^2} \tag{3.13}$$

and

$$\alpha_1 - \alpha_2 = \frac{\rho\Delta u}{a} \quad , \tag{3.14}$$

to give the following expressions for α_1 , α_2 and α_3 ,

$$\alpha_1 = \frac{1}{2a^2}(\Delta p + \rho a \Delta u) \tag{3.15a}$$

$$\alpha_2 = \frac{1}{2a^2}(\Delta p - \rho a \Delta u) \tag{3.15b}$$

and

$$\alpha_3 = \Delta\rho - \frac{\Delta p}{a^2} \quad . \tag{3.15c}$$

We have found α_1 , α_2 , α_3 such that

$$\Delta \underline{w} = \sum_{j=1}^{3} \alpha_j \underline{e}_j \tag{3.16}$$

to within $O(\Delta^2)$, and a routine calculation verifies that

$$\Delta \underline{F} = \sum_{j=1}^{3} \lambda_j \alpha_j \underline{e}_j \tag{3.17}$$

to within $O(\Delta^2)$. We are now in a position to construct the new approximate Riemann solver.

As in Roe [3], we consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$ and corresponding average eigenvectors $\tilde{\underline{e}}_1$, $\tilde{\underline{e}}_2$, $\tilde{\underline{e}}_3$ such that the relations (3.16) and (3.17) hold exactly for arbitrary states $\underline{w}_L$, $\underline{w}_R$ not necessarily close. Specifically, we seek averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{p}_i$, $\tilde{p}_\rho$, $\tilde{p}$ and $\tilde{i}$ in terms of two adjacent states $\underline{w}_L$, $\underline{w}_R$ such that

$$\Delta\underline{w} = \sum_{j=1}^{3} \tilde{\alpha}_j\tilde{\underline{e}}_j \qquad (3.18)$$

and

$$\Delta\underline{F} = \sum_{j=1}^{3} \tilde{\lambda}_j\tilde{\alpha}_j\tilde{\underline{e}}_j \ , \qquad (3.19)$$

where

$$\Delta(\bullet) = (\bullet)_R - (\bullet)_L \qquad (3.20a)$$

$$\underline{w} = (\rho,\rho u,e)^T \qquad (3.20b)$$

$$\underline{F}(\underline{w}) = (\rho u,p+\rho u^2,u(e+p))^T$$

$$e = \rho i + \tfrac{1}{2}\rho u^2 \qquad (3.20d)$$

$$p = p(\rho,i) \qquad (3.20e)$$

$$\tilde{\lambda}_{1,2,3} = \tilde{u} + \tilde{a},\ \tilde{u}-\tilde{a},\ \tilde{u} \qquad (3.21a)$$

$$\tilde{\underline{e}}_{1,2,3} = \begin{pmatrix} 1 \\ \tilde{u}+\tilde{a} \\ \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tilde{u}\tilde{a} \end{pmatrix}, \begin{pmatrix} 1 \\ \tilde{u}-\tilde{a} \\ \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \tilde{u}\tilde{a} \end{pmatrix}, \begin{pmatrix} 1 \\ \tilde{u} \\ \tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i} \end{pmatrix} \qquad (3.21b)$$

$$\tilde{a}_1 = \frac{1}{2\tilde{a}^2}(\Delta p + \tilde{\rho}\tilde{a}\Delta u)$$

$$\tilde{\alpha}_2 = \frac{1}{2\tilde{a}^2}(\Delta p - \tilde{\rho}\tilde{a}\Delta u) \tag{3.22b}$$

$$\tilde{\alpha}_3 = \Delta\rho - \frac{\Delta p}{\tilde{a}^2} \quad , \tag{3.22c}$$

and $\tilde{a}$ is given by

$$\tilde{\rho}\tilde{a}^2 = \frac{\tilde{p}\tilde{p}_i}{\tilde{\rho}} + \tilde{\rho}\tilde{p}_\rho \tag{3.23}$$

The problem of finding averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{p}_i$, $\tilde{p}_\rho$, $\tilde{p}$ and $\tilde{i}$ subject to equations (3.18)-(3.23) will subsequently be denoted by (*).
N.B. The quantities $\tilde{p}_i$ and $\tilde{p}_\rho$ denote approximations to the partial derivatives p_i and p_ρ , respectively.)

The solution of problem (*) will be sought in a similar way to that adopted by Roe [3] in the specialised, ideal gas case. We note, however, that problem (*) is equivalent to seeking an approximation to the Jacobian, A , namely $\tilde{A}$ with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\underline{\tilde{e}}_i$, which is an alternative approach also used in the ideal gas case by Roe [1].

The first step in the analysis of problem (*) is to write out equations (3.18) and (3.19) explicitly, namely,

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \tag{3.24a}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u}+\tilde{a}) + \tilde{\alpha}_2(\tilde{u}-\tilde{a}) + \tilde{\alpha}_3\tilde{u} \tag{3.24b}$$

$$\begin{aligned}\Delta e = \Delta(\rho i) + \Delta\left(\frac{\rho u^2}{2}\right) &= \tilde{\alpha}_1\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tilde{u}\tilde{a}\right) \\ &+ \tilde{\alpha}_2\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \tilde{u}\tilde{a}\right) \\ &+ \tilde{\alpha}_3\left(\tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}\right)\end{aligned} \tag{3.24c}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u}+\tilde{a}) + \tilde{\alpha}_2(\tilde{u}-\tilde{a}) + \tilde{\alpha}_3\tilde{u} \qquad (3.24d)$$

$$\Delta(p+\rho u^2) = \Delta p + \Delta(\rho u^2) = \tilde{\alpha}_1(\tilde{u}+\tilde{a})^2 + \tilde{\alpha}_2(\tilde{u}-\tilde{a})^2 + \tilde{\alpha}_3\tilde{u}^2 \qquad (3.24e)$$

and

$$\begin{aligned}\Delta\big(u(e+p)\big) &= \Delta(\rho u i) + \Delta\Big(\frac{\rho u^3}{2}\Big) + \Delta(up) \\ &= \tilde{\alpha}_1(\tilde{u}+\tilde{a})\Big(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 + \tilde{u}\tilde{a}\Big) \\ &+ \tilde{\alpha}_2(\tilde{u}-\tilde{a})\Big(\frac{\tilde{p}}{\tilde{\mu}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \tilde{u}\tilde{a}\Big) \\ &+ \tilde{\alpha}_3\tilde{u}\Big(\tilde{i} + \tfrac{1}{2}\tilde{u}^2 - \frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}\Big) \quad .\end{aligned} \qquad (3.24f)$$

Equation (3.24a) is satisfied by any average we care to define, while equation (3.24b) is the same as equation (3.24d); thus we have to solve equations (3.24c-f). From equation (3.24d) we have

$$\begin{aligned}\Delta(\rho u) &= \tilde{u}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + \tilde{a}(\tilde{\alpha}_1 - \tilde{\alpha}_2) \\ &= \tilde{u}\Delta\rho + \tilde{\rho}\Delta u \quad ,\end{aligned} \qquad (3.25)$$

and from equation (3.24e) we obtain

$$\begin{aligned}\Delta(\rho u^2) &= \tilde{u}^2(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + 2\tilde{u}\tilde{a}(\tilde{\alpha}_1 - \tilde{\alpha}_2) \\ &= \tilde{u}^2\Delta\rho + 2\tilde{u}\tilde{\rho}\Delta u \quad .\end{aligned} \qquad (3.26)$$

Substituting for $\tilde{\rho}$ from equation (3.25) into equation (3.26) yields the following quadratic equation for $\tilde{u}$

$$\tilde{u}^2\Delta\rho - 2\tilde{u}\Delta(\rho u) + \Delta(\rho u^2) = 0 \quad . \qquad (3.27)$$

Only one solution of equation (3.27) is productive, namely

$$\begin{aligned}\tilde{u} &= \frac{\Delta(\rho u) - \sqrt{(\Delta(\rho u))^2 - \Delta\rho\Delta(\rho u^2)}}{\Delta\rho} \\ &= \frac{\sqrt{\rho_L}\,u_L + \sqrt{\rho_R}\,u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \quad ,\end{aligned} \qquad (3.28)$$

which, on substituting $\tilde{u}$ into equation (3.25), gives

$$\tilde{\rho} = \frac{\Delta(\rho u) - \tilde{u}\Delta\rho}{\Delta u} = \sqrt{\rho_L \rho_R} \quad . \tag{3.29}$$

We have now determined $\tilde{\rho}$ and $\tilde{u}$, and with these we can show that

$$\Delta\left(\frac{\rho u^3}{2}\right) - \frac{\tilde{u}^3}{2}\Delta\rho - 3\,\frac{\tilde{\rho}u^2}{2}\,\Delta u = \frac{(\Delta u)^3 \tilde{\rho}^2}{2(\sqrt{\rho_L} + \sqrt{\rho_R})^2} \tag{3.30}$$

$$\Delta(up) - \tilde{u}\Delta p = \tilde{\rho}\Delta u \frac{\left(\sqrt{\rho_L}\,\frac{p_L}{\rho_L} + \sqrt{\rho_R}\,\frac{p_R}{\rho_R}\right)}{\sqrt{\rho_L} + \sqrt{\rho_R}} , \tag{3.31}$$

and

$$\frac{\sqrt{\rho_L}u_L^2 + \sqrt{\rho_R}u_R^2}{\sqrt{\rho_L} + \sqrt{\rho_R}} - \tilde{u}^2 = \frac{\tilde{\rho}(\Delta u)^2}{(\sqrt{\rho_L} + \sqrt{\rho_R})^2} , \tag{3.32}$$

all of which will be used later.

We are now left with equations (3.24c) and (3.24f). Before we deal with the general case, we focus attention on the consequences of equations (3.24c) and (3.24f) in the case of an ideal gas.

Using equations (3.22a)-(3.23), the difference between the left hand side and the right hand side of equation (3.24c), denoted by D, can be written

$$D = \Delta(\rho i) - \tilde{i}\Delta\rho - \tilde{\rho}\Delta i + \frac{\tilde{\rho}}{\tilde{p}_i}(\tilde{p}_i\Delta i + \tilde{p}_\rho\Delta\rho - \Delta p) . \tag{3.33}$$

Now, for an ideal gas $p = (\gamma-1)\rho i$, $p_i = (\gamma-1)\rho$ and $p_\rho = (\gamma-1)i$ so that

$$\tilde{p}_i = (\gamma-1)\tilde{\rho} \quad , \quad \tilde{p}_\rho = (\gamma-1)\tilde{i}$$

and

$$\frac{\tilde{\rho}}{\tilde{p}_i}(\tilde{p}_i\Delta i + \tilde{p}_\rho\Delta\rho - \Delta p) = \tilde{\rho}\Delta i + \tilde{i}\Delta\rho - \Delta(\rho i) ,$$

and thus $D = 0$, in this case. Therefore, for an ideal gas, equation (3.24c) is automatically satisfied and only equation (3.24f) remains. In particular, $\tilde{a}$ is determined from equation (3.23), i.e.

$$\tilde{a}^2 = (\gamma-1)\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i}\right) \tag{3.34}$$

and we now show that in this ideal gas case, $\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i}$ is completely determined in terms of $\rho_L, \rho_R, u_L, u_R, p_L$ and p_R . Furthermore, the third component of $\tilde{\underline{e}}_3$ reduces to $\frac{1}{2}\tilde{u}^2$. Now, using equations (3.22a)-(3.23), equation (3.24f) can be rewritten as

$$\Delta(\rho u i) - \tilde{u}\tilde{i}\Delta\rho - \tilde{\rho}\tilde{i}\Delta u + \Delta(up) - \tilde{u}\Delta p - \tilde{p}\Delta u$$

$$+ \Delta\left(\frac{\rho u^3}{2}\right) - \frac{\tilde{u}^3}{2}\Delta\rho - \frac{3}{2}\tilde{\rho}\tilde{u}^2\Delta u$$

$$+ \frac{\tilde{\rho}\tilde{u}}{\tilde{p}_i}(\tilde{p}_\rho\Delta\rho - \Delta p) = 0 , \tag{3.35}$$

and using the relationships $p = (\gamma-1)\rho i$, $\tilde{p}_i = (\gamma-1)\tilde{\rho}$, $\tilde{p}_\rho = (\gamma-1)\tilde{i}$ for an ideal gas, equation (3.35) becomes

$$\frac{\gamma}{\gamma-1}\left(\Delta(up) - \tilde{u}\Delta p\right) - \tilde{p}\Delta u - \tilde{\rho}\tilde{i}\Delta u$$

$$+ \Delta\left(\frac{\rho u^3}{2}\right) - \frac{\tilde{u}^3}{2}\Delta\rho - \frac{3}{2}\tilde{\rho}\tilde{u}^2\Delta u = 0 . \tag{3.36}$$

If we now use the identities given by equations (3.30)-(3.32) and divide throughout by $\tilde{\rho}\Delta u$, equation (3.36) yields

$$\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} = \frac{\gamma}{\gamma-1} \frac{\left(\sqrt{\rho_L}\frac{p_L}{\rho_L} + \sqrt{\rho_R}\frac{p_R}{\rho_R}\right)}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$+ \frac{\sqrt{\rho_L}u_L^2 + \sqrt{\rho_R}u_R^2}{2(\sqrt{\rho_L} + \sqrt{\rho_R})} - \frac{1}{2}\tilde{u}^2 , \tag{3.37}$$

which completely determines $\tilde{a}$, since from equation (3.34) $\tilde{a}^2 = (\gamma-1)\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i}\right)$. In addition, if we define a mean enthalpy $\tilde{H} = \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \frac{1}{2}\tilde{u}^2$ and note that $\frac{\gamma p}{\rho(\gamma-1)} = \frac{p}{\rho} + i$, then equation (3.37) states that

$$\tilde{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} . \tag{3.38}$$

and so

$$\tilde{a}^2 = (\gamma - 1)(\tilde{H} - \tfrac{1}{2}\tilde{u}^2) . \tag{3.39}$$

The results are in accordance with those similarly derived by Roe [3].

We now return to the general case and endeavour to retain certain results found in the ideal case.

We begin by rewriting equations (3.24c) and (3.24f), using equations (3.22a)-(3.23), to give

$$\Delta(\rho i) - \tilde{i}\Delta\rho - \frac{\tilde{p}\Delta p}{\tilde{\rho}\tilde{a}^2} + \tilde{\alpha}_3\tilde{\rho}\frac{\tilde{p}_\rho}{\tilde{p}_i} = 0 \tag{3.40}$$

and

$$\Delta(\rho u i) - \tilde{u}\tilde{i}\Delta\rho - \tilde{\rho}\tilde{i}\Delta u + \Delta(up) - \tilde{u}\Delta p - \tilde{p}\Delta u$$

$$+ \Delta\left(\frac{\rho}{2}u^3\right) - \frac{\tilde{u}^3}{2}\Delta\rho - \frac{3}{2}\tilde{\rho}\tilde{u}^2\Delta u$$

$$- \frac{\tilde{u}\tilde{p}\Delta p}{\tilde{\rho}\tilde{a}^2} + \tilde{\alpha}_3\tilde{u}\tilde{\rho}\frac{\tilde{p}_\rho}{\tilde{p}_i} = 0 . \tag{3.41}$$

Now, subtracting equation (3.40) multiplied by $\tilde{u}$ from equation (3.41) and using equations (3.30)-(3.32) together with the following identity

$$\Delta(\rho u i) - \tilde{u}\Delta(\rho i) = \tilde{\rho}\Delta u \frac{\left(\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R\right)}{\sqrt{\rho_L} + \sqrt{\rho_R}} ,$$

we obtain, after division by $\rho\Delta u$,

$$\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 = \frac{\sqrt{\rho_L}\left(\frac{p_L}{\rho_L} + i_L + \frac{1}{2}u_L^2\right) + \sqrt{\rho_R}\left(\frac{p_R}{\rho_R} + i_R + \frac{1}{2}u_R^2\right)}{\sqrt{\rho_L} + \sqrt{\rho_R}} . \quad (3.42)$$

Therefore, if we define a mean enthalpy, $\tilde{H}$, by

$$\tilde{H} = \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \tfrac{1}{2}\tilde{u}^2 , \quad (3.43)$$

we find, from equation (3.42), that

$$\tilde{H} = \frac{\sqrt{\rho_L}\, H_L + \sqrt{\rho_R}\, H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} , \quad (3.44)$$

as in the ideal case. We have now specified $\tilde{\rho}$, $\tilde{u}$, $\frac{\tilde{p}}{\tilde{\rho}} + \tilde{i}$: thus, in order to specify $\tilde{p}_i$, $\tilde{p}_\rho$, $\tilde{i}$ (and hence $\tilde{p}$), we focus our attention on equation (3.40) which can be written as

$$\Delta(\rho i) - \tilde{i}\Delta\rho - \tilde{\rho}\Delta i + \frac{\tilde{\rho}}{\tilde{p}_i}\left(\tilde{p}_i\Delta i + \tilde{p}_\rho\Delta\rho - \Delta p\right) = 0 . \quad (3.45)$$

A number of choices can now be made, but it is clear that the most natural choice is to take

$$\Delta(\rho i) - \tilde{i}\Delta\rho - \tilde{\rho}\Delta i = 0 , \quad (3.46)$$

i.e.

$$\tilde{i} = \frac{\Delta(\rho i) - \rho\Delta i}{\Delta\rho} = \frac{\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} , \tag{3.47}$$

in which case (3.45) gives

$$\Delta p = \tilde{p}_i \Delta i + \tilde{p}_\rho \Delta\rho \tag{3.48}$$

as a necessary condition. Therefore, all we need to complete our approximate Riemann solver is to choose approximations $\tilde{p}_i$, $\tilde{p}_\rho$ to p_i, p_ρ such that (3.48) holds. This is a straightforward matter to achieve as we now see.

We propose approximations $\tilde{p}_i$, $\tilde{p}_\rho$ to p_i and p_ρ as follows

$$\tilde{p}_i = \begin{cases} \frac{1}{\Delta i}\left(\frac{1}{2}\left[p(\rho_R, i_R) + p(\rho_L, i_R)\right] - \frac{1}{2}\left[p(\rho_R, i_L) + p(\rho_L, i_L)\right]\right) & \text{if } \Delta i \neq 0 \quad (3.49a) \\ \frac{1}{2}\left[p_i(\rho_L, i) + p_i(\rho_R, i)\right] & \text{if } \Delta i = 0, i_L = i_R = i \quad (3.49b) \end{cases}$$

and

$$\tilde{p}_\rho = \begin{cases} \frac{1}{\Delta\rho}\left(\frac{1}{2}\left[p(\rho_R, i_R) + p(\rho_R, i_L)\right] - \frac{1}{2}\left[p(\rho_L, i_R) + p(\rho_L, i_L)\right]\right) & \text{if } \Delta\rho \neq 0 \quad (3.50a) \\ \frac{1}{2}\left[p_\rho(\rho, i_L) + p_\rho(\rho, i_R)\right] & \text{if } \Delta\rho = 0,\ \rho_L = \rho_R = \rho . \quad (3.50b) \end{cases}$$

It is a simple matter to check that, for each of the combinations arising from the approximations given by equations (3.49a)-(3.50b), equation (3.48) is satisfied. In particular, if the equation of state consists of a series of terms of the form $p = \mathcal{R}(\rho) I(i)$ where $\mathcal{R}$, I depend on ρ, i, respectively, then equations (3.49a)-(3.50b) become

$$\tilde{p}_i = \begin{cases} \bar{\mathcal{R}} \dfrac{\Delta I}{\Delta i} & \text{if} \quad \Delta i \neq 0 \qquad (3.51a) \\ \bar{\mathcal{R}}\, I'(i) & \text{if} \quad \Delta i = 0 \ , \qquad (3.51b) \\ & \qquad i_L = i_R = i \ , \end{cases}$$

and

$$\tilde{p}_\rho = \begin{cases} \bar{I} \dfrac{\Delta \mathcal{R}}{\Delta \rho} & \text{if} \quad \Delta \rho \neq 0 \qquad (3.52a) \\ \bar{I}\, \mathcal{R}'(\rho) & \text{if} \quad \Delta \rho = 0 \ , \qquad (3.52b) \\ & \qquad \rho_L = \rho_R = \rho \ , \end{cases}$$

where $\Delta(\circ) = (\circ)_R - (\circ)_L$ as before, and $\bar{\circ} = \frac{1}{2}[(\circ)_L + (\circ)_R]$, the arithmetic mean. Finally, we must note that if we are dealing with an ideal gas, then $p = (\gamma-1)\rho i$ where γ is constant, and we replace equations (3.49a)-(3.50b) by

$$\tilde{p}_i = (\gamma - 1)\tilde{\rho} \qquad (3.53a)$$

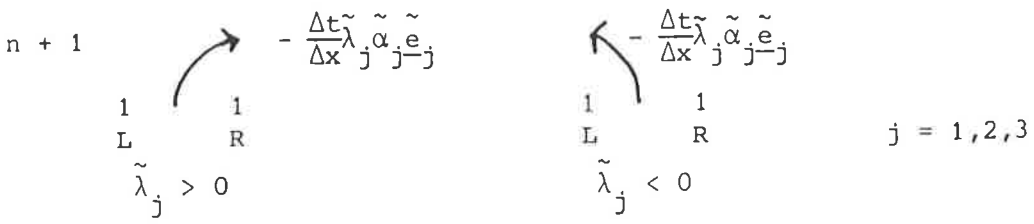
and

$$\tilde{p}_\rho = (\gamma - 1)\tilde{i} \ , \qquad (3.53b)$$

where $\tilde{\rho}$, $\tilde{i}$ are given by equations (3.29), (3.47), respectively.

Summarising, we can now apply our one-dimensional Riemann solver for the Euler equations with a general equation of state in a similar way to that of Roe [3] as follows.

Suppose at time level n we have data $\underline{w}_L$, $\underline{w}_R$ given at either end of the cell (x_L, x_R) , then we update $\underline{w}$ to time level $n+1$ in an upwind manner. Schematically, we increment $\underline{w}$:



where $\Delta x = x_R - x_L$, Δt is the time interval from level n to $n + 1$, and $\tilde{\lambda}_j$, $\tilde{\alpha}_j$, $\tilde{\underline{e}}_j$ are given by

$$\lambda_{1,2,3} = \tilde{u} + \tilde{a},\ \tilde{u} - \tilde{a},\ \tilde{u}$$

$$\tilde{\underline{e}}_{1,2,3} = \begin{pmatrix} 1 \\ \tilde{u} + \tilde{a} \\ \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \frac{1}{2}\tilde{u}^2 \\ + \tilde{u}\,\tilde{a} \end{pmatrix},\ \begin{pmatrix} 1 \\ \tilde{u} - \tilde{a} \\ \frac{\tilde{p}}{\tilde{\rho}} + \tilde{i} + \frac{1}{2}\tilde{u}^2 \\ - \tilde{u}\,\tilde{a} \end{pmatrix},\ \begin{pmatrix} 1 \\ \tilde{u} \\ \tilde{i} + \frac{1}{2}\tilde{u}^2 - \tilde{\rho}\frac{\tilde{p}_\rho}{\tilde{p}_i} \end{pmatrix}$$

$$\tilde{\alpha}_{1,2,3} = \frac{1}{2\tilde{a}^2}(\Delta p + \tilde{\rho}\tilde{a}\Delta u),\ \frac{1}{2\tilde{a}^2}(\Delta p - \tilde{\rho}\tilde{a}\Delta u),\ \Delta\rho - \frac{\Delta p}{\tilde{a}^2}$$

$$\tilde{\rho} = \sqrt{\rho_L \rho_R}\ , \qquad \tilde{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{i} = \frac{\sqrt{\rho_L}i_L + \sqrt{\rho_R}i_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}\ , \qquad \tilde{H} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}$$

$$\tilde{p} = \tilde{\rho}(\tilde{H} - \tilde{i} - \tfrac{1}{2}\tilde{u}^2)\ , \qquad \tilde{a} = \frac{\tilde{p}\tilde{p}_i}{\tilde{\rho}^2} + \tilde{p}_\rho\ ,$$

$\tilde{p}_i$, $\tilde{p}_\rho$ are given by equations (3.49a)-(3.50b) and $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$. In addition, we can use the idea of flux limiters [4] to create a second order algorithm which is oscillation free, and we can modify the scheme to disperse entropy violating solutions, (see [5]).

The Riemann solver we have constructed in this section is a conservative algorithm, and has the important shock capturing property guaranteed by equations (3.18)-(3.19). In the next section we give examples of different equations of state.

4. EQUATIONS OF STATE

In this section we give three different forms of the equation of state for a fluid.

(a) Ideal gas equation of state

This can be written in the general form

$$p = (\gamma - 1)\rho i \tag{4.1}$$

where γ is a constant and represents the ratio of specific heat capacities of the fluid. Typical values for γ are $\gamma = \frac{5}{3}$ for a monatomic gas, e.g. helium, and $\gamma = 1.4$ for a diatomic gas, e.g. air.

(b) Stiffened equation of state

This is usually written in the form

$$p = B\left(\frac{\rho}{\rho_0} - 1\right) + (\gamma - 1)\rho i \tag{4.2}$$

where B is a constant, and ρ_0 represents a reference density. This form of the equation of state is a simple extension of the ideal gas equation, and as such can be used in test problems originally designed for ideal gases.

(c) General equation of state

A more general equation of state has been developed by R.K. Osborne at the Los Alamos Scientific Laboratory, and can be written in the form

$$p = [1/(E + \phi_0)]\{\zeta(a_1 + a_2|\zeta|)$$

$$+ E[b_0 + \zeta(b_1 + b_2\zeta) + E(c_0 + c_1\zeta)]\} \tag{4.3}$$

where $E = \rho_0 i$, $\zeta = \frac{\rho}{\rho_0} - 1$ and the constants ρ_0, a_1, a_2, b_0, b_1, b_2,

c_0, c_1, ϕ_0 depend on the material in question. Typical values for the material constants for copper are given in §6.

Our algorithm requires knowledge of the derivatives p_i, p_ρ which can be explicitly calculated in each of the three cases (a), (b), (c).

The most general equations of state may be presented in tabular form, but provided that data is available for p, p_i and p_ρ , we can always apply our algorithm as in cases (a)-(c).

In the next section we describe a standard test problem for the Euler equations with a general equation of state, and derive the exact solution.

5. A TEST PROBLEM

In this section we describe a standard test problem in gas dynamics and, using the Rankine-Hugoniot shock relations, we derive the exact solution to this problem.

The test problem we look at is concerned with shock reflection in one dimension of a gas governed by the Euler equations with a general equation of state. We consider a region $0 \leqq x \leqq 1$ with initial conditions (at $t = 0$) ,

$$\left.\begin{aligned} \rho &= \rho_0 \\ u &= -u_0 \\ i &= i_0 \end{aligned}\right\} , \qquad (5.1)$$

where $p_0 = p(\rho_0, i_0)$ is given. This represents a gas of constant density and pressure moving towards $x = 0$. The boundary $x = 0$ is a rigid wall and the exact solution describes shock reflection from the wall. The gas is brought to rest at $x = 0$ and, denoting initial values by (0) , pre-shocked values by (-) , and post-shocked values by (+) , we can postulate an exact solution of the form

$$\rho = \rho^+ , \quad u = u^+ = 0 , \quad i = i^+ , \quad (p = p^+ = p(\rho^+, i^+)) \quad \text{for } \frac{x}{t} < S \qquad (5.2a)$$

$$\rho = \rho^- , \quad u = u^- = -u_0 , \quad i = i^- = i_0 , \quad (p = p^- = p_0 = p(\rho_0, i_0)) \quad \text{for } \frac{x}{t} > S , \qquad (5.2b)$$

where the shock moves out from the origin with speed S , and $S, \rho^+, i^+, p^+ = p(\rho^+, i^+)$ are given by the Rankine-Hugoniot shock relations. Thus

$$S = \frac{[\rho u]}{[\rho]} = \frac{[p + \rho u^2]}{[\rho u]} = \frac{[u(e + p)]}{[e]} \qquad (5.3)$$

where $[v] = v^+ - v^-$ denotes the jump in v across the shock. We therefore have to solve equations (5.3) for S, ρ^+, i^+, p^+ subject to the initial conditions given by equation (5.1), and a precise form for the equation of state $p = p(\rho,i)$.

If we write out equations (5.3), using equations (5.1)-(5.2b), we obtain

$$S = \frac{\rho_0 u_0}{\rho - \rho_0} \qquad (5.4a)$$

$$S = \frac{p - p_0 - \rho_0 u_0^2}{\rho_0 u_0} \qquad (5.4b)$$

and

$$S = u_0 \frac{(\rho_0 i_0 + \frac{1}{2}\rho_0 u_0^2 + p_0)}{\rho i - \rho_0 i_0 - \frac{1}{2}\rho_0 u_0^2} \qquad (5.4c)$$

where ρ, u, i, p denote ρ^+, u^+, i^+, p^+ for simplicity, and p_0, p are given by

$$p = f(\rho,i) \qquad (5.4d)$$

$$p_0 = f(\rho_0,i_0) \qquad (5.4e)$$

(N.B. The function f is used to denote the particular equation of state and satisfies $f(\rho_0,0) = 0$, i.e. $p_0 = 0$ when $i_0 = 0$.)

Since it is easily shown that equation (5.4c) can be deduced from equations (5.4a-b), we concern ourselves only with the solution of equations (5.4a-b), (5.4d-e). From equation (5.4a) it is possible to show that

$$H_0 - i = \frac{p_0}{\rho} \tag{5.5}$$

and from equation (5.4b) we can show that

$$(p - p_0 - \rho_0 u_0^2)\left(\frac{\rho}{\rho_0} - 1\right) - \rho_0 u_0^2 = 0 . \tag{5.6}$$

The solution of equations (5.5)-(5.6) now splits into two cases (i) $i_0 = 0$ and (ii) $i_0 \neq 0$.

Case (i) $\underline{i_0 = 0}$

If $i_0 = 0$, so $p_0 = f(\rho_0, 0) = 0$ then from equation (5.5)

$$i = H_0 = \frac{p_0}{\rho_0} + i_0 + \tfrac{1}{2} u_0^2 = \tfrac{1}{2} u_0^2 \tag{5.7}$$

and equation (5.6) becomes

$$\left(f(\rho, \tfrac{1}{2} u_0^2) - \rho_0 u_0^2\right)\left(\frac{\rho}{\rho_0} - 1\right) - \rho_0 u_0^2 = 0 , \tag{5.8}$$

which can be solved iteratively, using the method of bisection, for any function f . Thus, we obtain a value for $\rho^+ = \rho$. and from equation (5.7) we have that $i^+ = i = \frac{1}{2} u_0^2$. Finally, we find $p^+ = p = f(\rho, i)$ and the shock speed

$$s = \rho_0 u_0 / (\rho - \rho_0) .$$

Case (ii) $\underline{i_0 \neq 0}$

If $i_0 \neq 0$ and $p_0 = f(\rho_0, i_0) \neq 0$ then from equation (5.5)

$$i = H_0 - \frac{p_0}{\rho} = \frac{p_0}{\rho_0} + i_0 - \tfrac{1}{2} u_0^2 - \frac{p_0}{\rho} , \tag{5.9}$$

and substituting this expression for i into equation (5.6) we obtain

$$\left(f(\rho,\frac{p_0}{\rho_0} + \tfrac{1}{2}u_0^2 + i_0 - \frac{p_0}{\rho}) - \rho_0 u_0^2\right)\left(\frac{\rho}{\rho_0} - 1\right) - \rho_0 u_0^2 = 0 \ , \qquad (5.10)$$

which can be solved iteratively for ρ , using the method of bisection, for any function f . Thus we obtain $\rho^+ = \rho$, and from equation (5.9) we have that $i^+ = i = \frac{p_0}{\rho_0} + i_0 + \frac{1}{2}u_0^2 - \frac{p_0}{\rho}$. Finally, we find $p^+ = p = f(\rho,i)$ and $S = \rho_0 u_0/(\rho - \rho_0)$.

A special case that can be solved exactly is when the equation of state is of the form

$$p = f(\rho,i) = \alpha\rho + \beta\rho i + \delta \ , \qquad (5.11)$$

and hence

$$p - p_0 = \alpha(\rho - \rho_0) + \beta(\rho i - \rho_0 i_0) \ . \qquad (5.12)$$

From equation (5.5) we find that

$$\rho i - \rho_0 i_0 = (\rho - \rho_0)H_0 + \frac{\rho_0 u_0^2}{2} \ , \qquad (5.13)$$

and using equations (5.12)-(5.13) equation (5.6) gives the following quadratic equation for $\frac{\rho}{\rho_0} - 1$

$$\frac{(\alpha + \beta H_0)}{u_0^2}\left(\frac{\rho}{\rho_0} - 1\right)^2 + \left(\frac{\beta}{2} - 1\right)\left(\frac{\rho}{\rho_0} - 1\right) - 1 = 0 \ . \qquad (5.14)$$

Now, from equation (5.11) we can write

$$\frac{\alpha + \beta H_0}{u_0^2} = \frac{(\beta + 1)p_0}{\rho_0 u_0^2} - \frac{\delta}{\rho_0 u_0^2} + \frac{\beta}{2} \ .$$

Therefore, the positive solution of equation (5.14) is

$$\frac{\rho}{\rho_0} = \frac{2z + 1 + \frac{\beta}{2} + \sqrt{(\frac{\beta}{2} + 1)^2 + 4z}}{2z + \beta}$$

where

$$z = \frac{(\beta + 1)p_0}{\rho_0 u_0^2} - \frac{\delta}{\rho_0 u_0^2} .$$

Thus, we obtain a value for $\rho^+ = \rho$, and from equation (5.9) we have that $i = i^+ = \frac{p_0}{\rho_0} + i_0 + \frac{1}{2} u_0^2 - \frac{p_0}{\rho}$. Finally, we find $p^+ = p = \alpha\rho + \beta\rho i + \delta$, and the shock speed $S = \rho_0 u_0/(\rho-\rho_0)$.

In this section we have described a means of obtaining the exact solution to the shock reflection problem for a general equation of state, while for the particular equation of state given by equation (5.11) we have given the solution explicitly. To obtain the solution in the case of a stiffened equation of state we set $\alpha = \frac{B}{\rho_0}$, $\beta = \gamma - 1$ and $\delta = - B$, and for the ideal gas equation of state we set $\alpha = \delta = 0$, $\beta = \gamma - 1$.

In the next section we give the numerical results obtained for the test problem described here.

6. NUMERICAL RESULTS

In this section we show the numerical results obtained for the test problem given in §5 using the Riemann solver described in §3.

Each of the figures refers to one of the equations of state given in §4 with different values of the parameters and initial conditions.

(a) Ideal equation of state

We take $\gamma = \frac{5}{3}$ and $\gamma = 1.4$ with the initial data

$$\rho(x,0) = \rho_0 = 1$$

$$u(x,0) = -u_0 = -1$$

and choose $i(x,0) = i_0$ such that the pressure jump across the shock, i.e. p^+/p^-, takes the values ∞, 10 or 2 .

(b) Stiffened equation of state

The parameters and initial data are taken to have the same values as for (a) and we choose $B = 1.0$. Three pressure ratios are obtained as for (a).

(c) General equation of state for Copper

We consider the general equation of state given by equation (4.3) with values for the parameters corresponding to Copper, i.e.

$$\rho_0 = 8.90 , \quad a_1 = 4.9578 , \quad a_2 = 3.6884 ,$$

$$b_0 = 7.4727 , \quad b_1 = 11.519 , \quad b_2 = 5.5251 ,$$

$$c_0 = 0.39493 , \quad c_1 = 0.52883 , \quad \phi_0 = 3.6000$$

together with the initial data

$$\rho(x,0) = \rho_0 = 8.9$$

$$u(x,0) = -u_0$$

where u_0 = 0.4, 0.7 or 1.0. Again we choose $i(x,0) = i_0$ such that the pressure ratio p^+/p^- takes the three values ∞ , 10 or 2.

In each case we take 100 mesh points in $0 \leq x \leq 1$, and choose the output time so that the shock has moved a distance of 0.3. All computations have been done using the 'Superbee limiter', (see [4]).

Figure 1	Equation (a)	$\gamma = \frac{5}{3}$	$p^+/p^- = \infty$
Figure 2	Equation (a)	$\gamma = \frac{5}{3}$	$p^+/p^- = 10$
Figure 3	Equation (a)	$\gamma = \frac{5}{3}$	$p^+/p^- = 2$
Figure 4	Equation (a)	$\gamma = 1.4$	$p^+/p^- = \infty$
Figure 5	Equation (a)	$\gamma = 1.4$	$p^+/p^- = 10$
Figure 6	Equation (a)	$\gamma = 1.4$	$p^+/p^- = 2$
Figure 7	Equation (b)	$\gamma = \frac{5}{3}$, B = 1.0	$p^+/p^- = \infty$
Figure 8	Equation (b)	$\gamma = \frac{5}{3}$, B = 1.0	$p^+/p^- = 10$
Figure 9	Equation (b)	$\gamma = \frac{5}{3}$, B = 1.0	$p^+/p^- = 2$
Figure 10	Equation (b)	$\gamma = 1.4$, B = 1.0	$p^+/p^- = \infty$
Figure 11	Equation (b)	$\gamma = 1.4$, B = 1.0	$p^+/p^- = 10$
Figure 12	Equation (b)	$\gamma = 1.4$, B = 1.0	$p^+/p^- = 2$
Figure 13	Equation (c)	$u_0 = 0.4$	$p^+/p^- = \infty$
Figure 14	Equation (c)	$u_0 = 0.4$	$p^+/p^- = 10$
Figure 15	Equation (c)	$u_0 = 0.4$	$p^+/p^- = 2$
Figure 16	Equation (c)	$u_0 = 0.7$	$p^+/p^- = \infty$

Figure 17	Equation (c)	$u_0 = 0.7$	$p^+/p^- = 10$
Figure 18	Equation (c)	$u_0 = 0.7$	$p^+/p^- = 2$
Figure 19	Equation (c)	$u_0 = 1.0$	$p^+/p^- = \infty$
Figure 20	Equation (c)	$u_0 = 1.0$	$p^+/p^- = 10$
Figure 21	Equation (c)	$u_0 = 1.0$	$p^+/p^- = 2$

In each case we can see that the approximate solution gives a good representation of the exact solution, in particular the correct shock speed has been achieved.

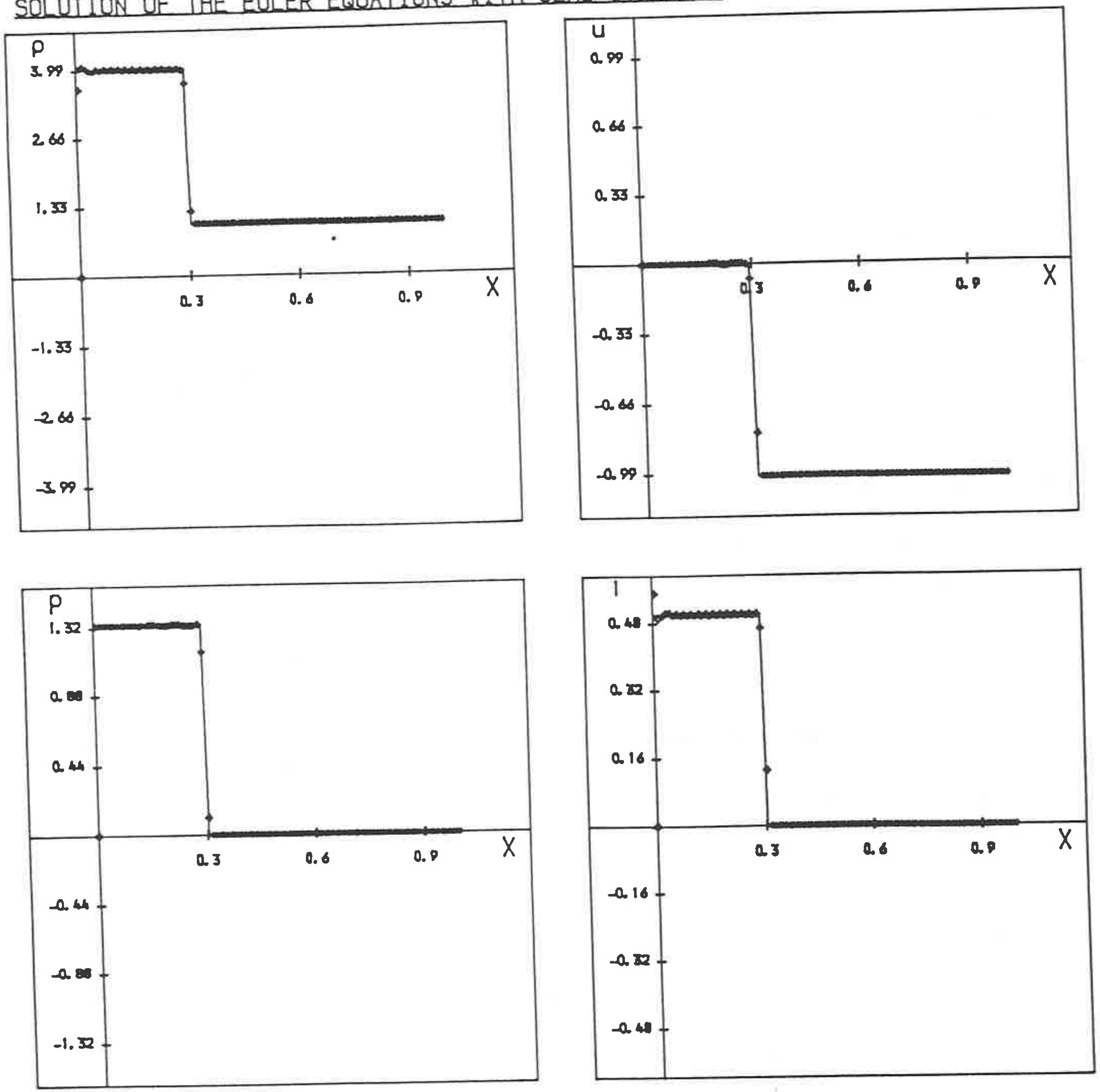
Finally, we compare the c.p.u. time to compute the results obtained for the ideal gas case (a) using (i) Roe's original Riemann solver, and (ii) our general Riemann solver applied to the ideal gas case. (N.B. Although (i) and (ii) are mathematically equivalent, (ii) is for the general case and would therefore expect to be more costly). The comparison, using an Amdahl V7, is as follows:-

(i) Using 'superbee' and 100 mesh points takes 0.0142 c.p.u. seconds to compute one time step, and a total of 1.6 c.p.u. seconds to reach a real time of 0.9 seconds using 112 time steps.

(ii) Using 'superbee' and 100 mesh points takes 0.0178 c.p.u. seconds to compute one time step, and a total of 2.0 c.p.u. seconds to reach a real time of 0.9 seconds using 112 time steps.

This shows that our general Riemann solver is only slightly more expensive than Roe's original, as was to be expected.

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
ı - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

Ideal equation of state :
$\gamma = 5/3$
100 Mesh points
224 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0040$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 0.000$
$(\imath = 0.000)$

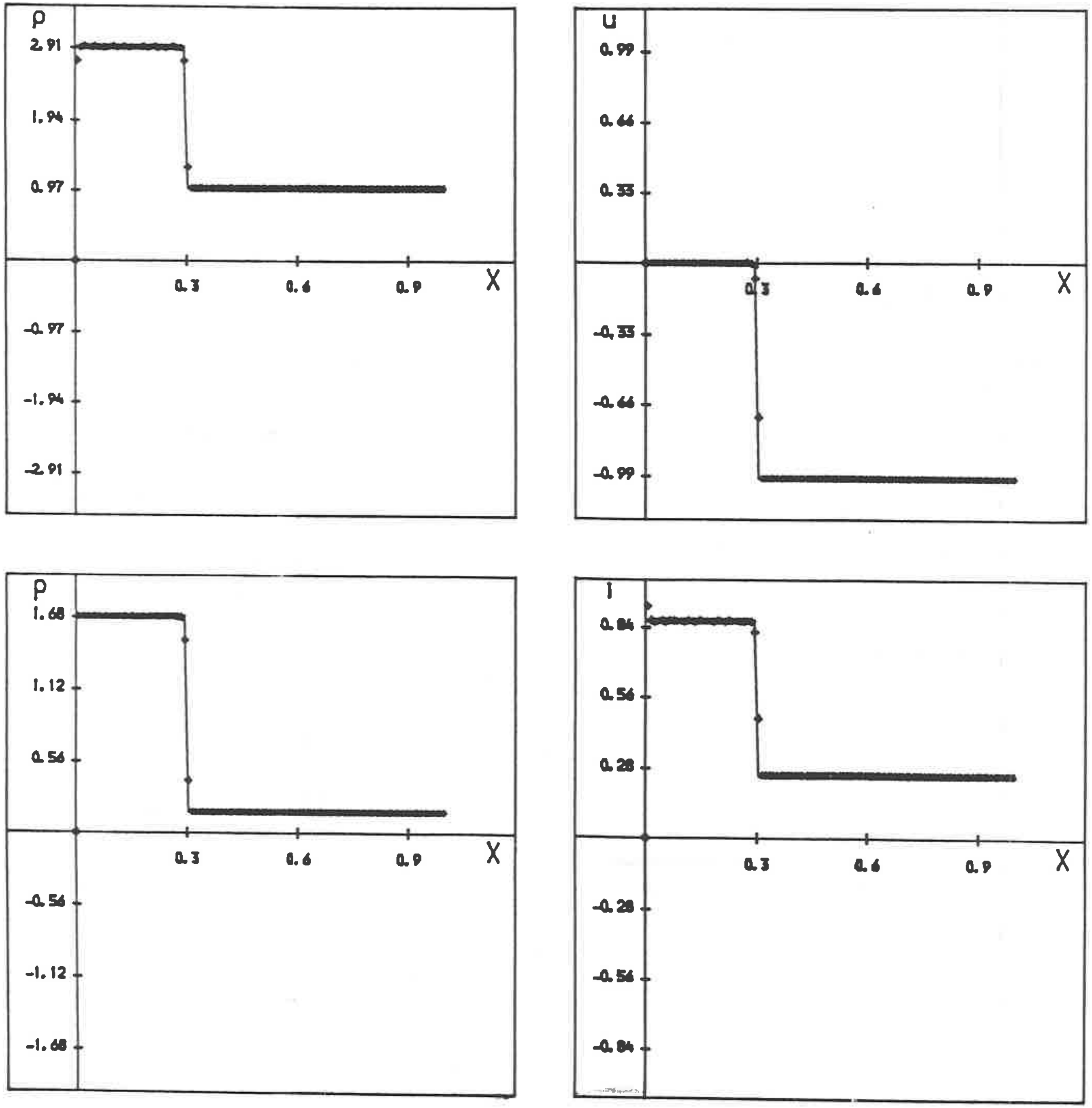
0 1

Reflected Boundary Conditions
at x = 0

0.896

Figure 1

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

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p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

Ideal equation of state :
$\gamma = 5/3$
100 Mesh points
221 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0026$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 1.000
u = -1.000
p = 0.169
(I = 0.253)

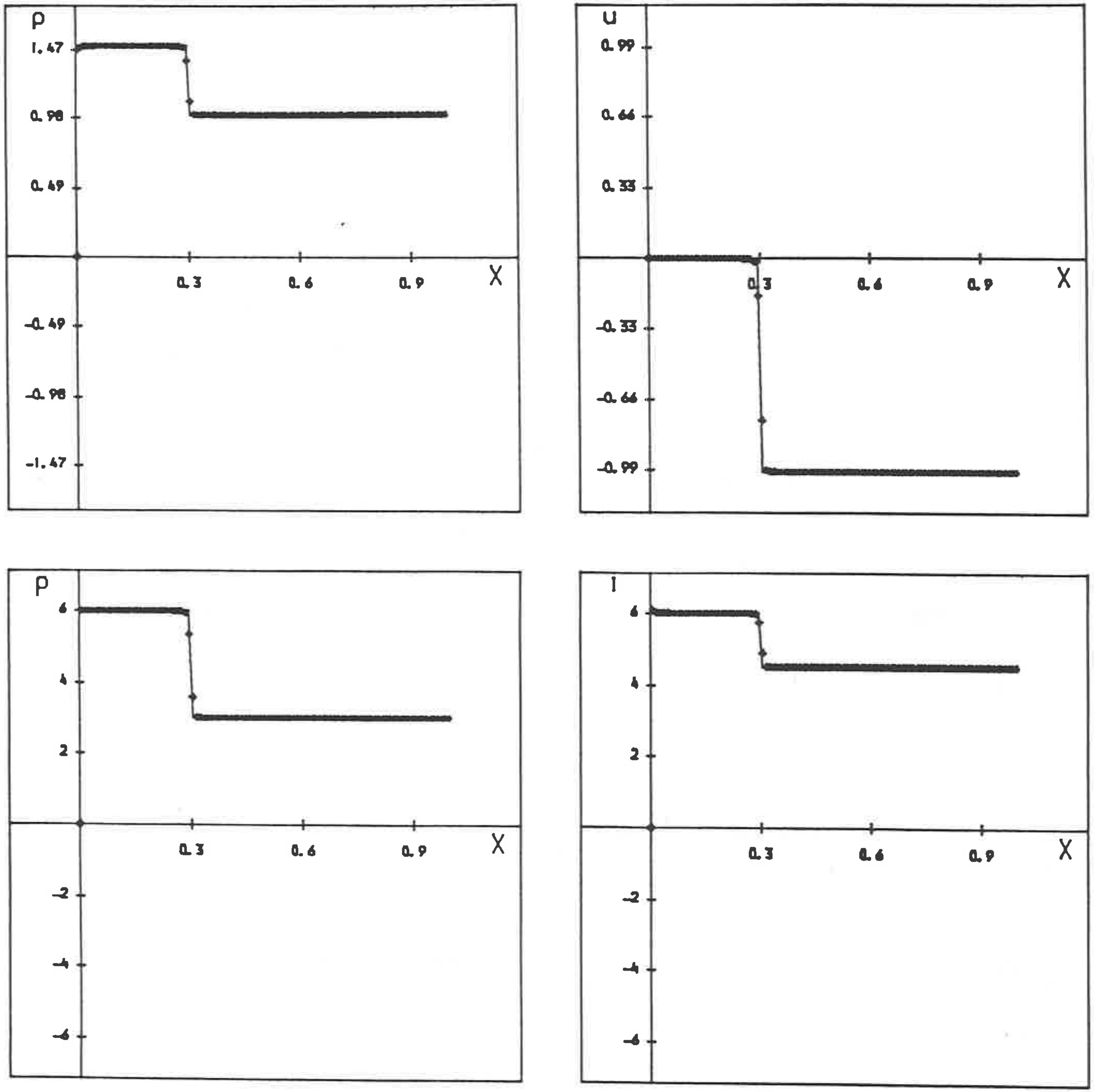
0 1

Reflected Boundary Conditions at x = 0

at time t = 0.578

Figure 2

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

Ideal equation of state :
$\gamma = 5/3$
100 Mesh points
121 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0012$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 3.000$
$(I = 4.500)$

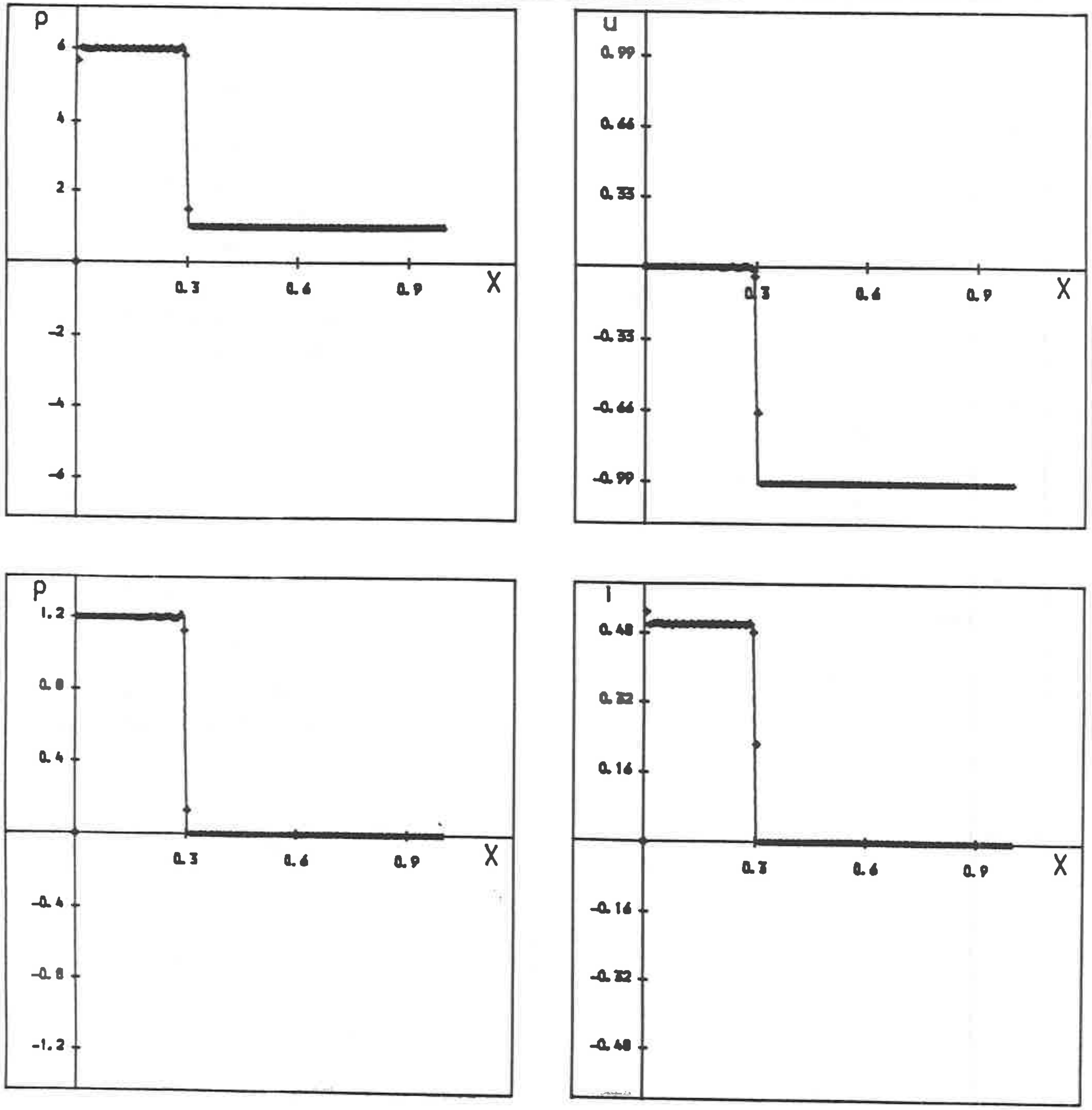
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.150

Figure 3

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

Ideal equation of state :
$\gamma = 1.4$
100 Mesh points
375 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0040$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 0.000$
$(I = 0.000)$

0 1

Reflected Boundary Conditions
at $x = 0$

at time t = 1.500

Figure 4

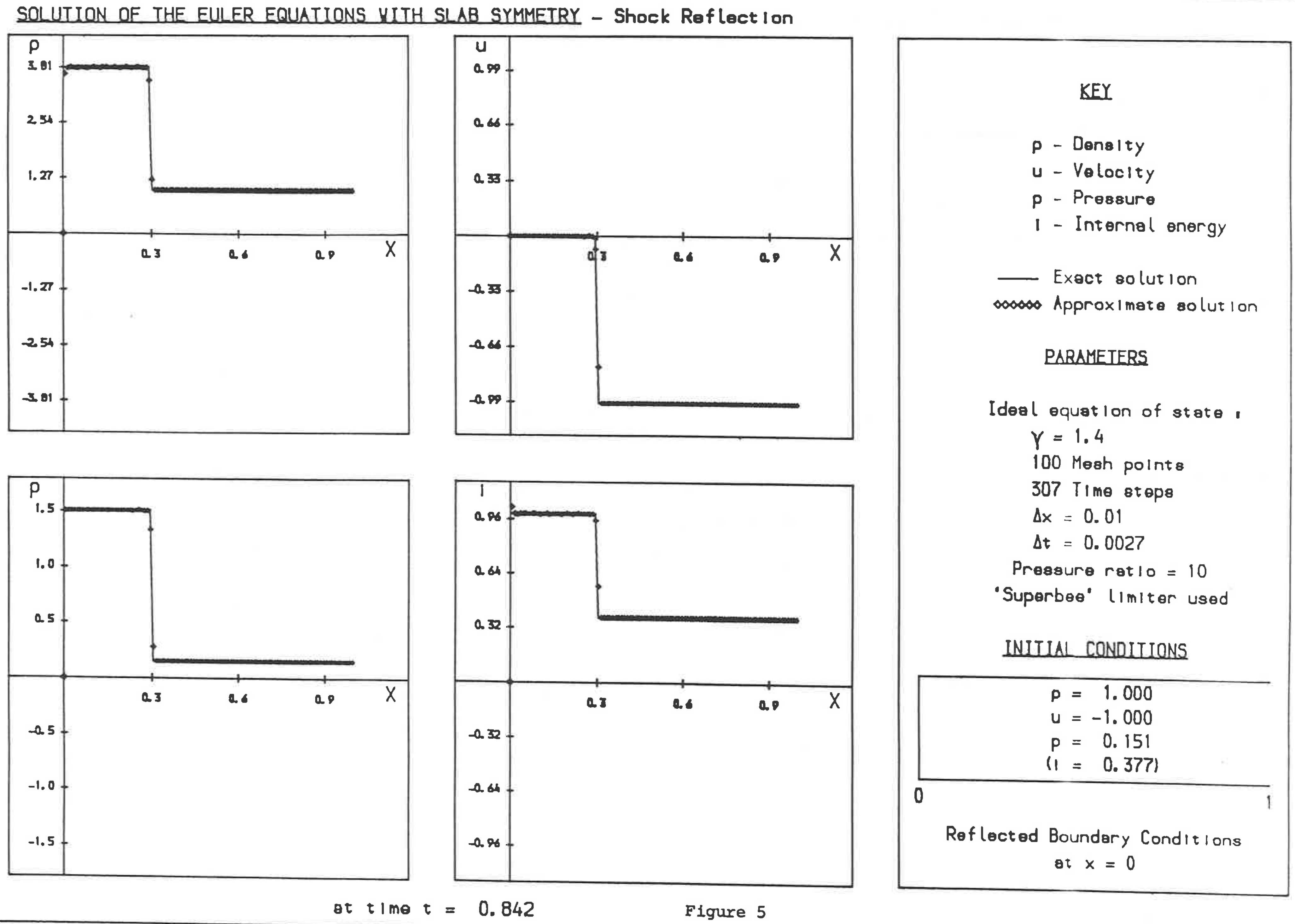
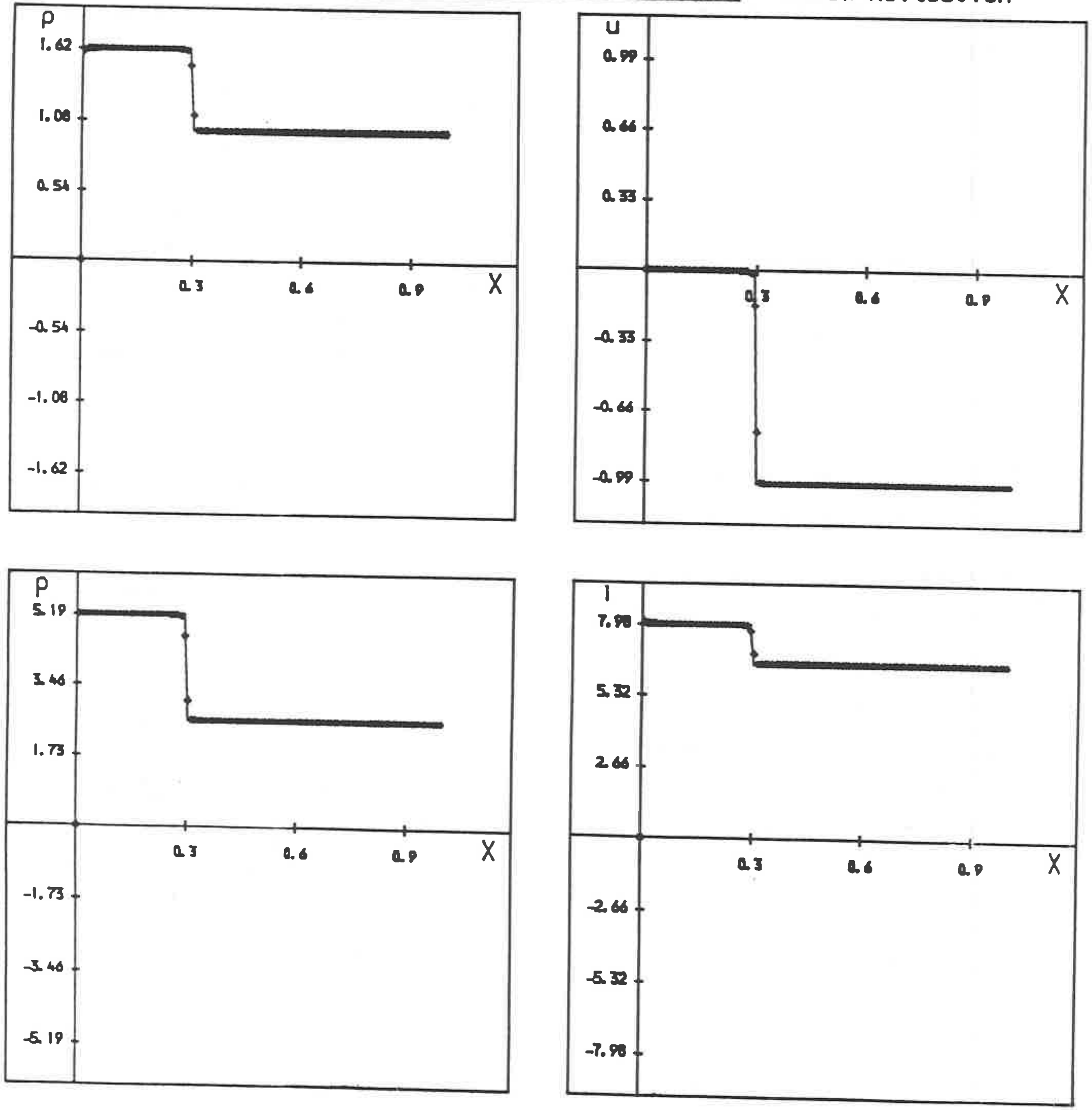

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection
ρ
3.81
2.54
1.27
-1.27
-2.54
-3.81
0.3
0.6
0.9
X
u
0.99
0.66
0.33
-0.33
-0.66
-0.99
p
1.5
1.0
0.5
-0.5
-1.0
-1.5
I
0.96
0.64
0.32
-0.32
-0.64
-0.96
KEY
ρ - Density
u - Velocity
p - Pressure
I - Internal energy
—— Exact solution
◊◊◊◊◊◊ Approximate solution
PARAMETERS
Ideal equation of state :
γ = 1.4
100 Mesh points
307 Time steps
Δx = 0.01
Δt = 0.0027
Pressure ratio = 10
'Superbee' limiter used
INITIAL CONDITIONS
ρ = 1.000
u = -1.000
p = 0.151
(I = 0.377)
0
1
Reflected Boundary Conditions
at x = 0
at time t = 0.842

Figure 5

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇ Approximate solution

PARAMETERS

Ideal equation of state :
$\gamma = 1.4$
100 Mesh points
136 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0014$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 1.000
u = -1.000
p = 2.600
(I = 6.500)

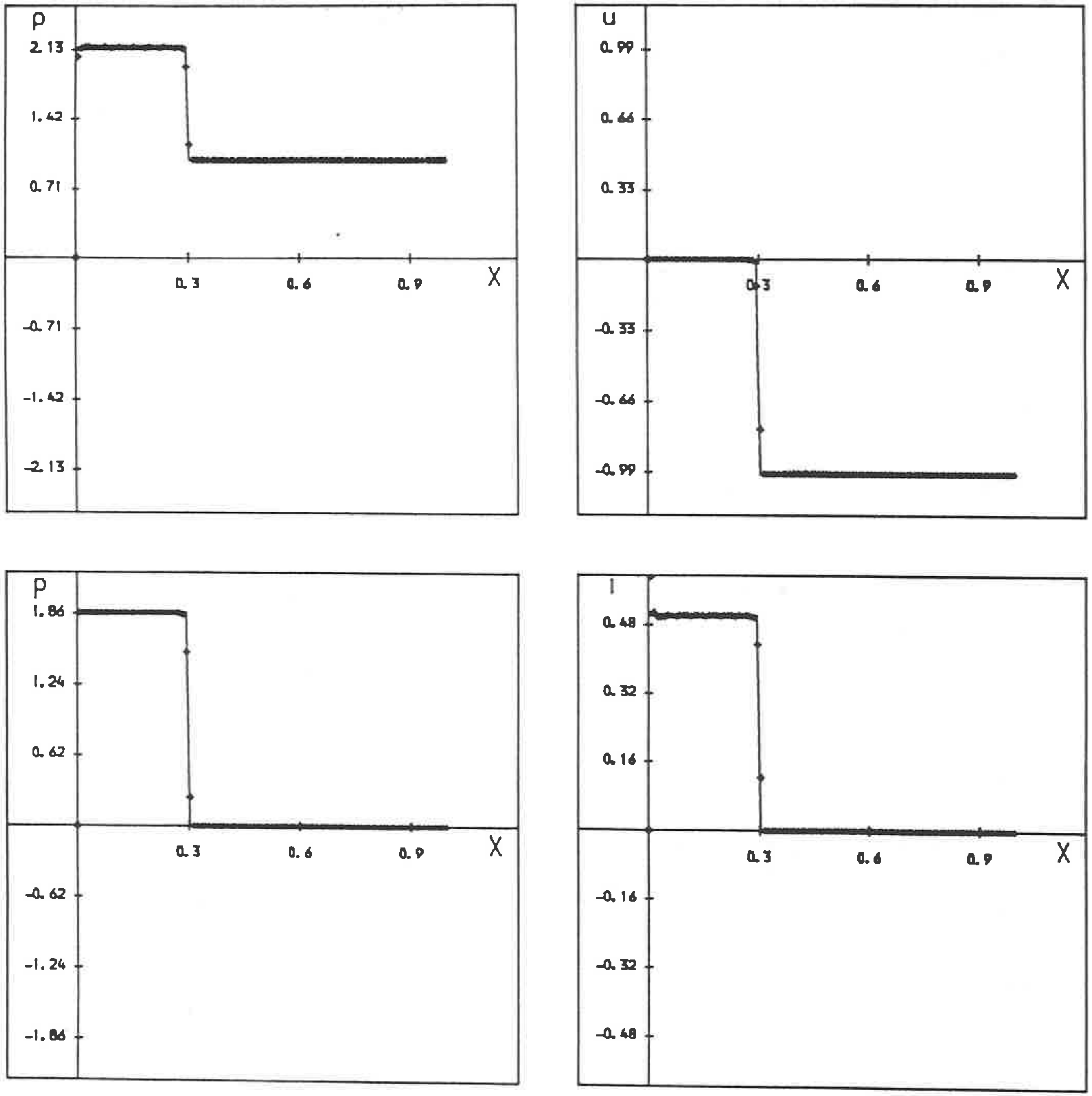
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.187

Figure 6

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

p - Density
u - Velocity
p - Pressure
i - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

Stiffened equation of state :
$\gamma = 5/3$, B = 1.00
100 Mesh points
172 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0020$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

p = 1.000
u = -1.000
p = 0.000
(i = 0.000)

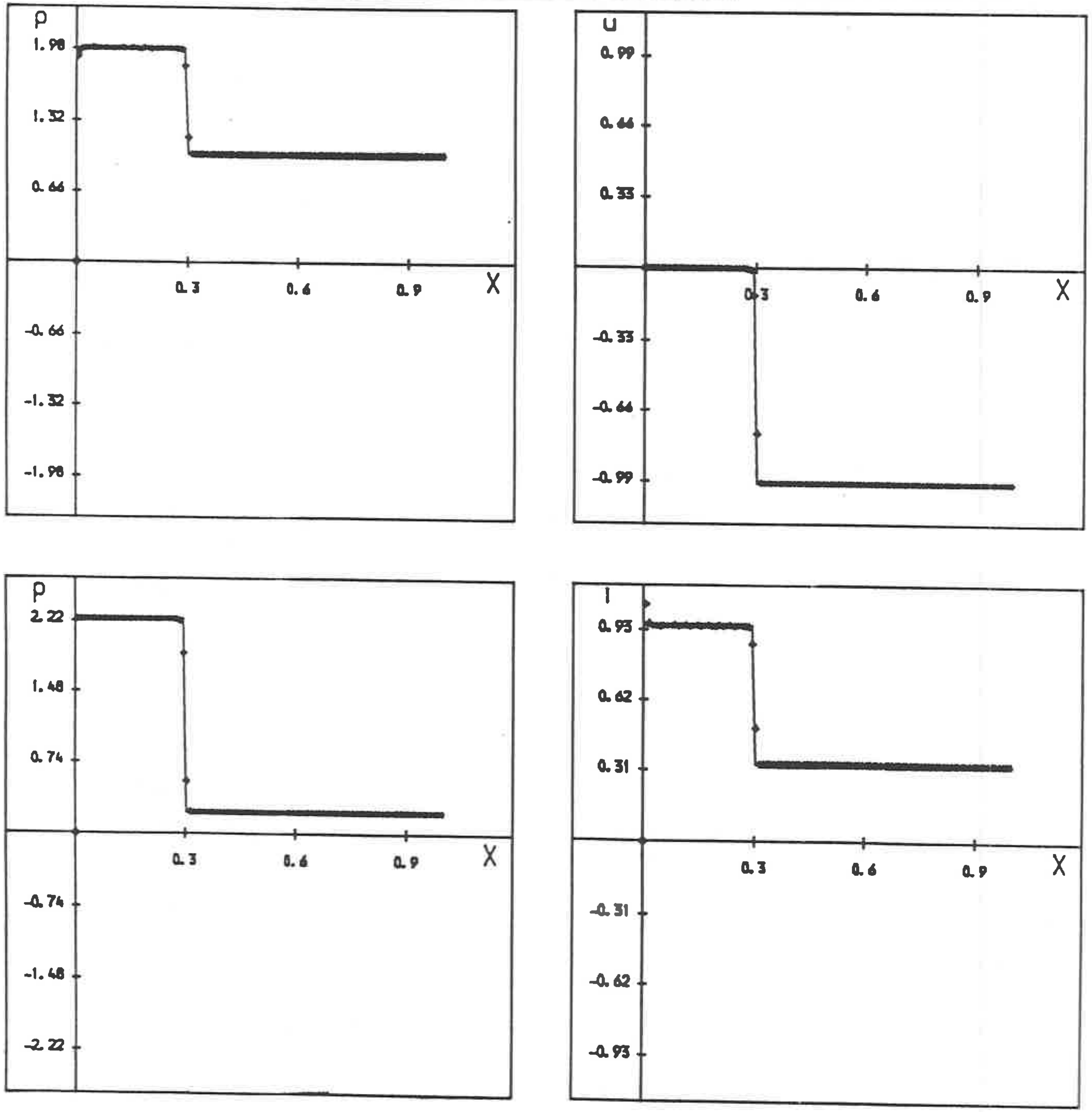
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.344

Figure 7

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

Stiffened equation of state :
$\gamma = 5/3$, $B = 1.00$
100 Mesh points
160 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0018$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 0.224$
$(I = 0.336)$

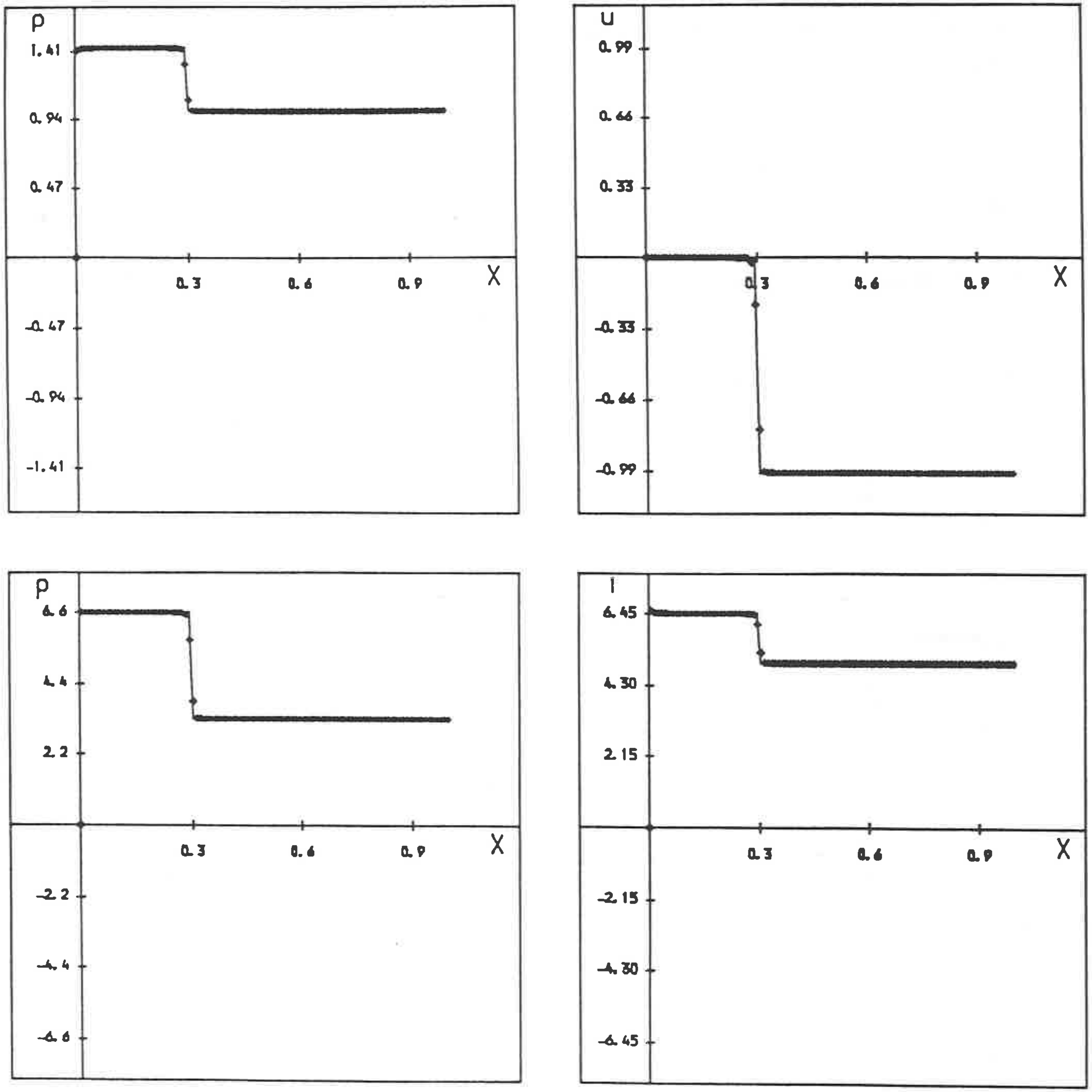
0 1

Reflected Boundary Conditions
at $x = 0$

at time t = 0.295

Figure 8

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇ Approximate solution

PARAMETERS

Stiffened equation of state :
$\gamma = 5/3$, B = 1.00
100 Mesh points
115 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0011$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 3.303$
$(I = 4.954)$

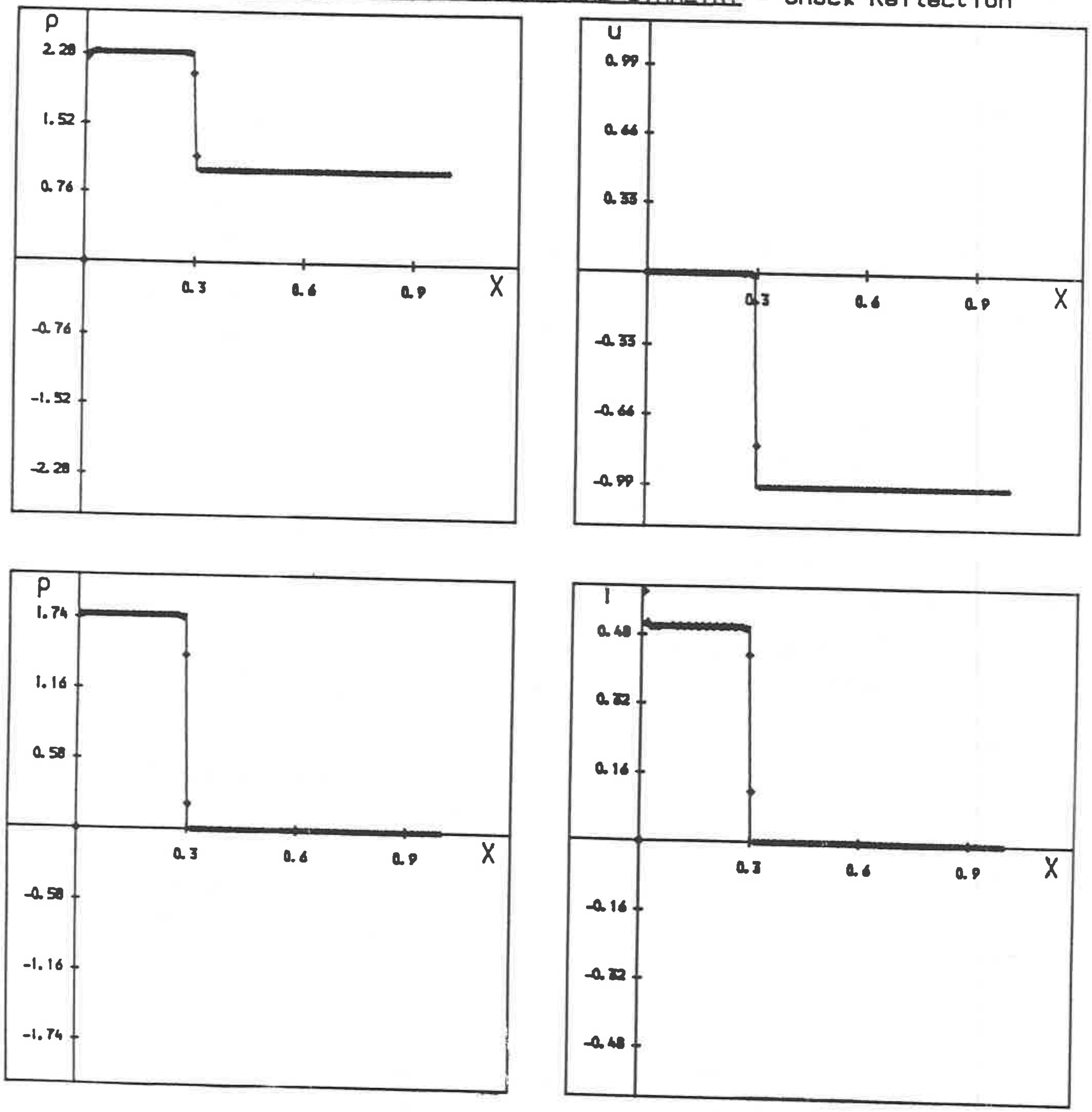
0 … 1

Reflected Boundary Conditions
at x = 0

at time t = 0.130

Figure 9

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

Stiffened equation of state :
$\gamma = 1.4$, $B = 1.00$
100 Mesh points
195 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0020$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 1.000
u = -1.000
p = 0.000
(I = 0.000)

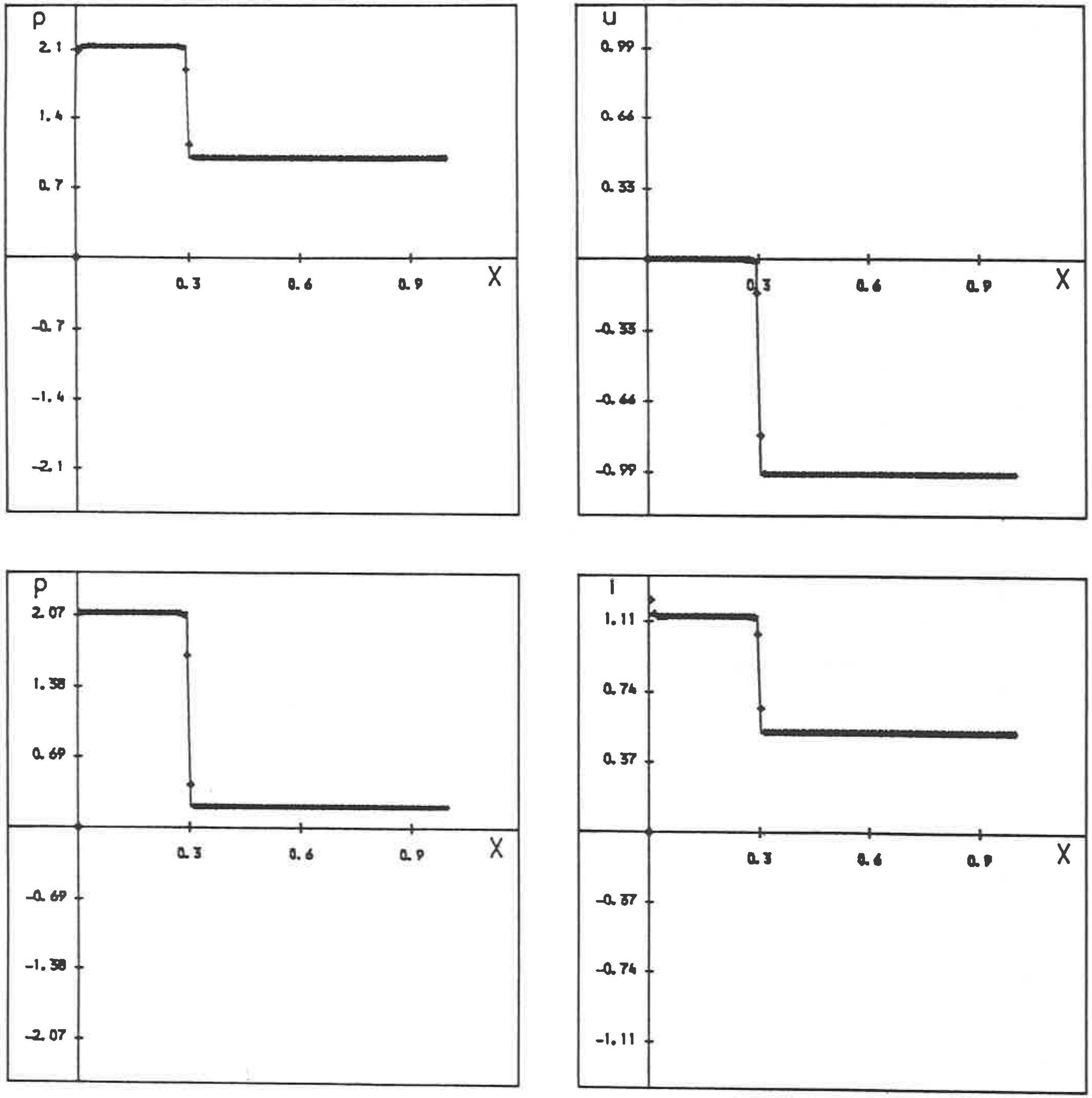
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.390

Figure 10

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

Stiffened equation of state :
$\gamma = 1.4$, $B = 1.00$
100 Mesh points
180 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0019$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 1.000$
$u = -1.000$
$p = 0.210$
$(I = 0.524)$

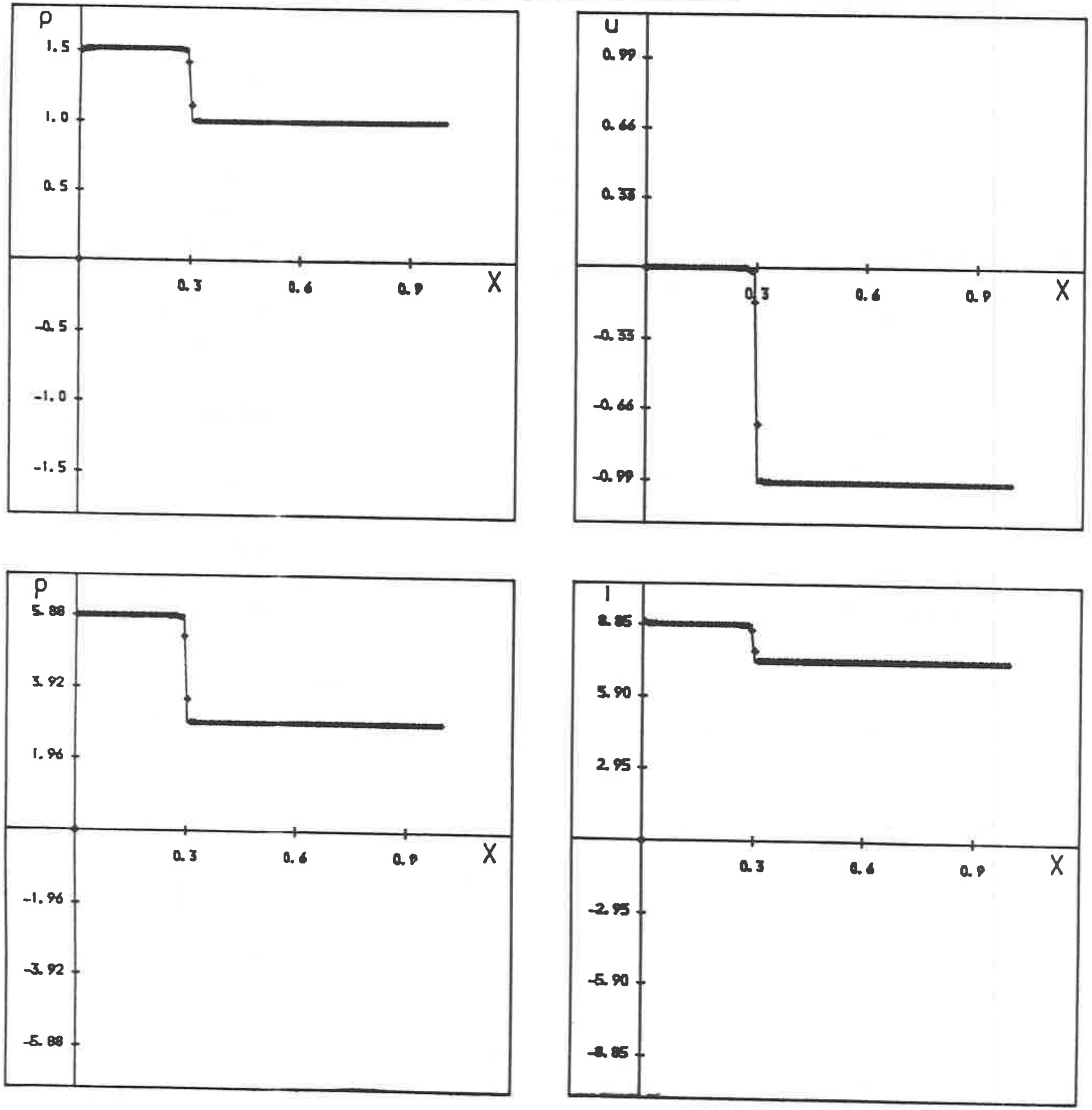
0 1

Reflected Boundary Conditions
at $x = 0$

at time t = 0.337

Figure 11

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

Stiffened equation of state ,
$\gamma = 1.4$, $B = 1.00$
100 Mesh points
126 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0012$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 1.000
u = -1.000
p = 2.940
(I = 7.350)

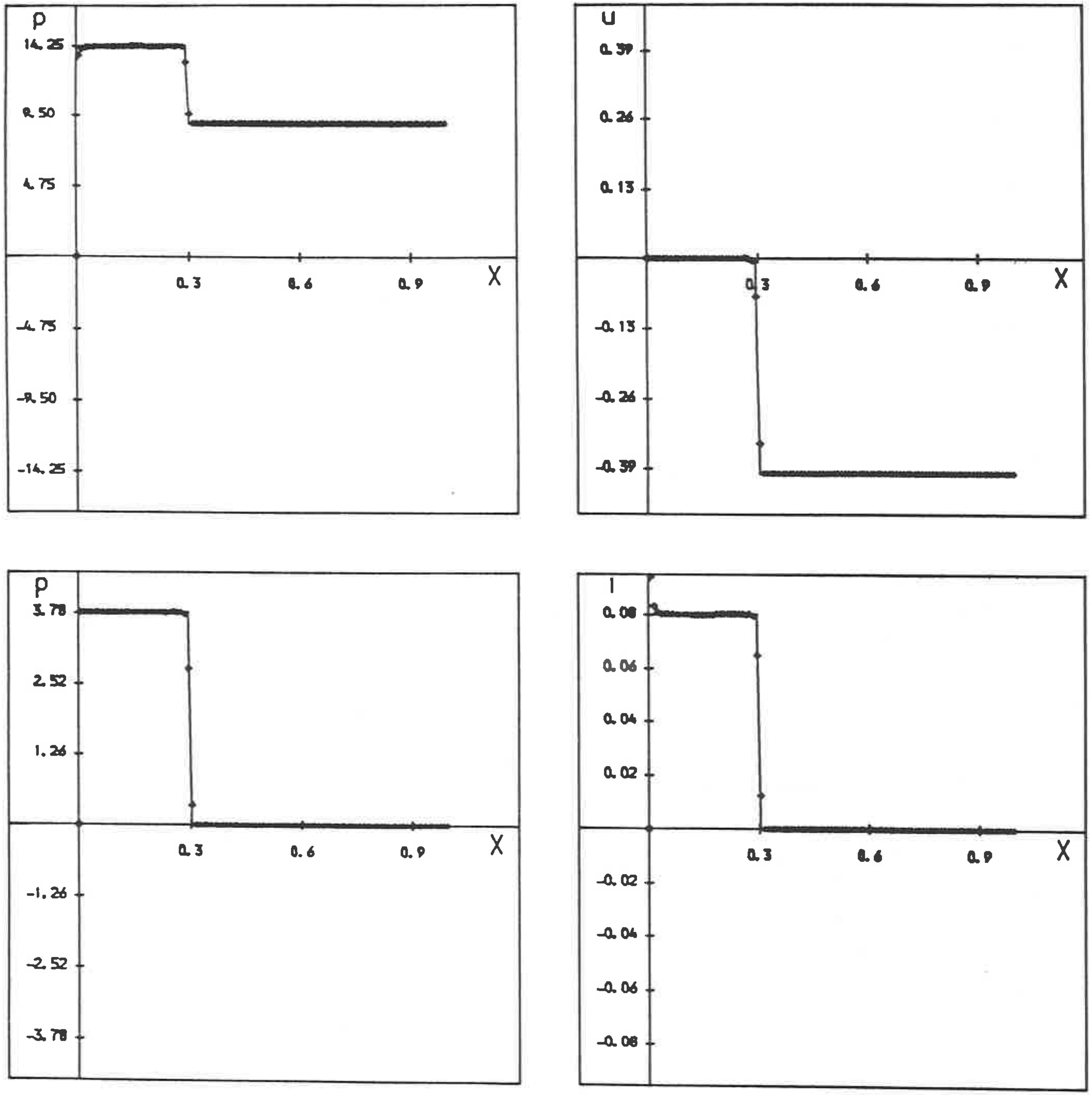
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.155

Figure 12

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state ;
for Copper due to R.K.Osborne
100 Mesh points
118 Time steps
Δx = 0.01
Δt = 0.0038
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

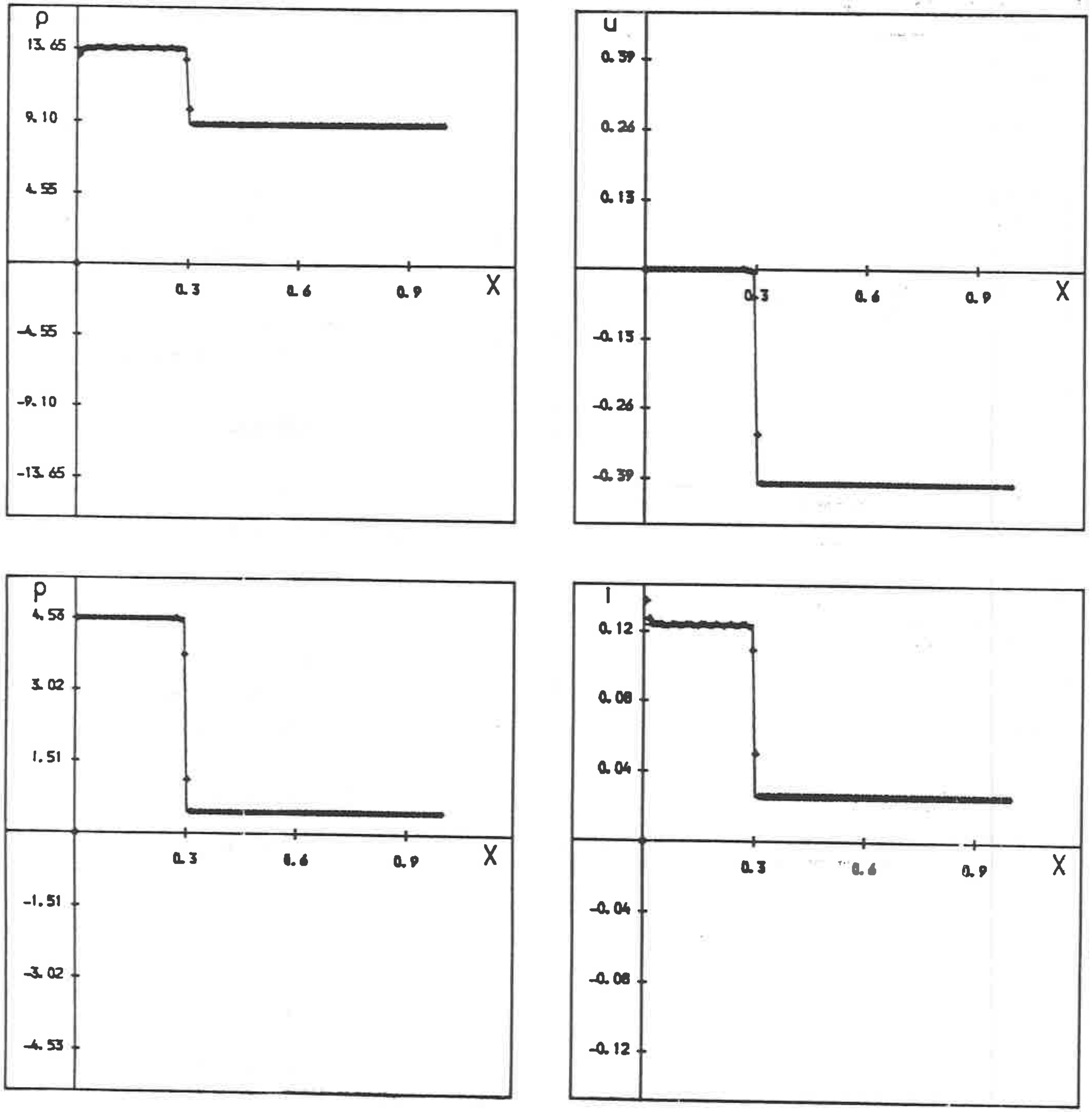
ρ = 8.900
u = -0.400
p = 0.000
(I = 0.000)

0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.447 Figure 13

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state ;
for Copper due to R.K.Osborne
100 Mesh points
112 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0036$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

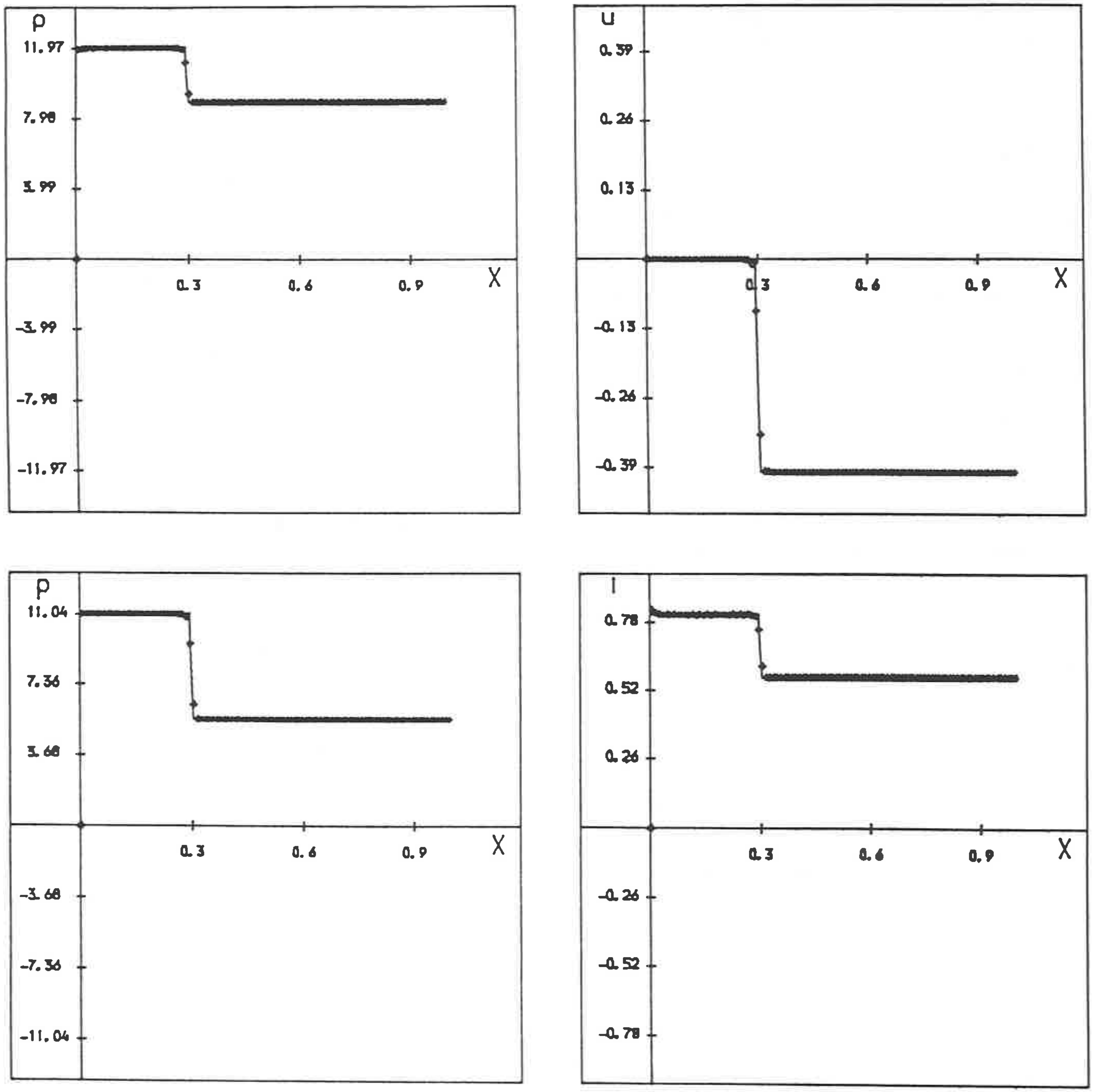
ρ = 8.900
u = -0.400
p = 0.453
(I = 0.026)

0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.401 Figure 14

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

——— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state :
for Copper due to R.K. Osborne
100 Mesh points
103 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0025$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 8.900$
$u = -0.400$
$p = 5.528$
$(I = 0.568)$

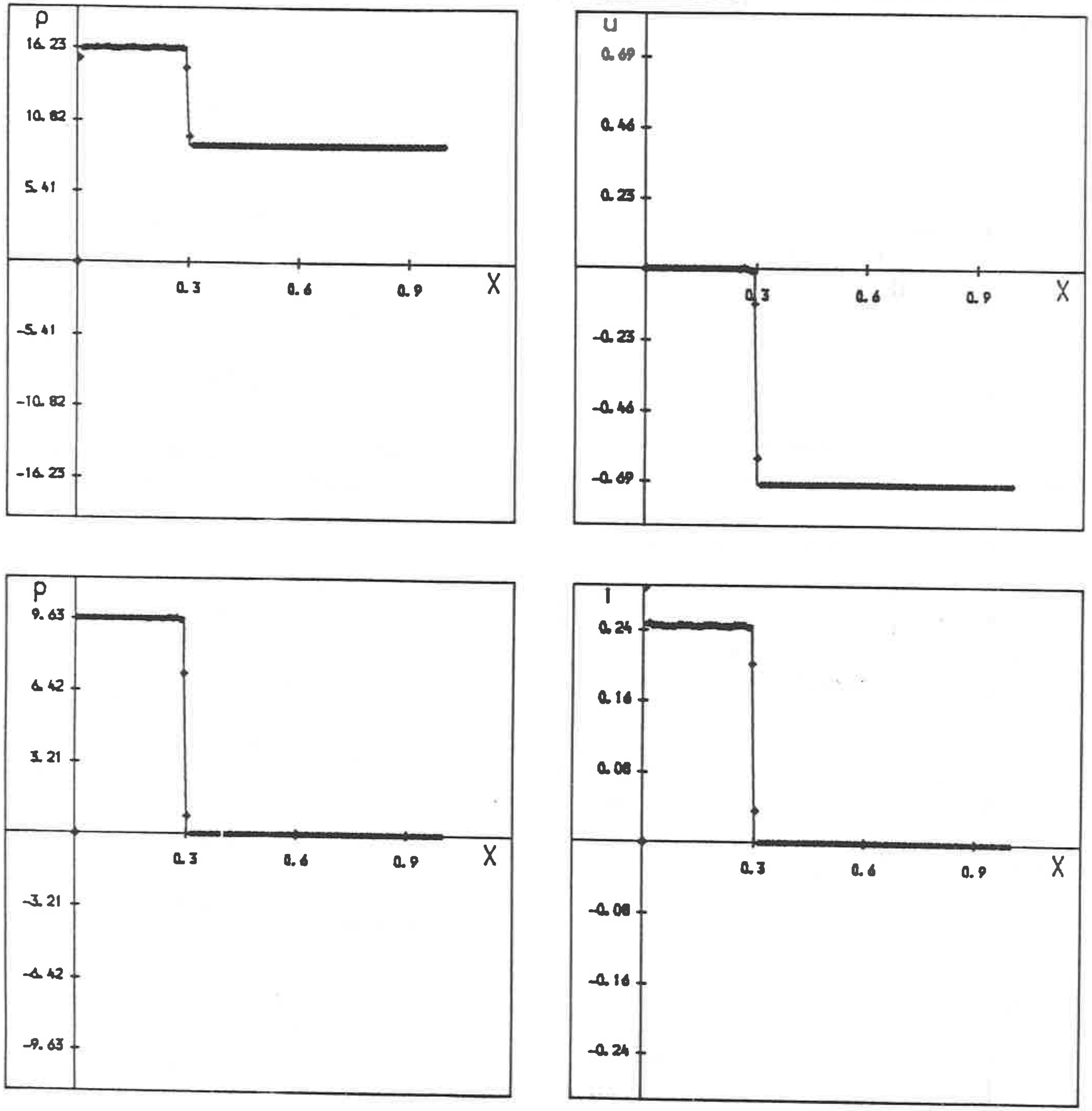
0 1

Reflected Boundary Conditions
at $x = 0$

at time $t = 0.259$

Figure 15

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

General equation of state ,
for Copper due to R.K. Osborne
100 Mesh points
123 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0029$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 8.900
u = -0.700
p = 0.000
(I = 0.000)

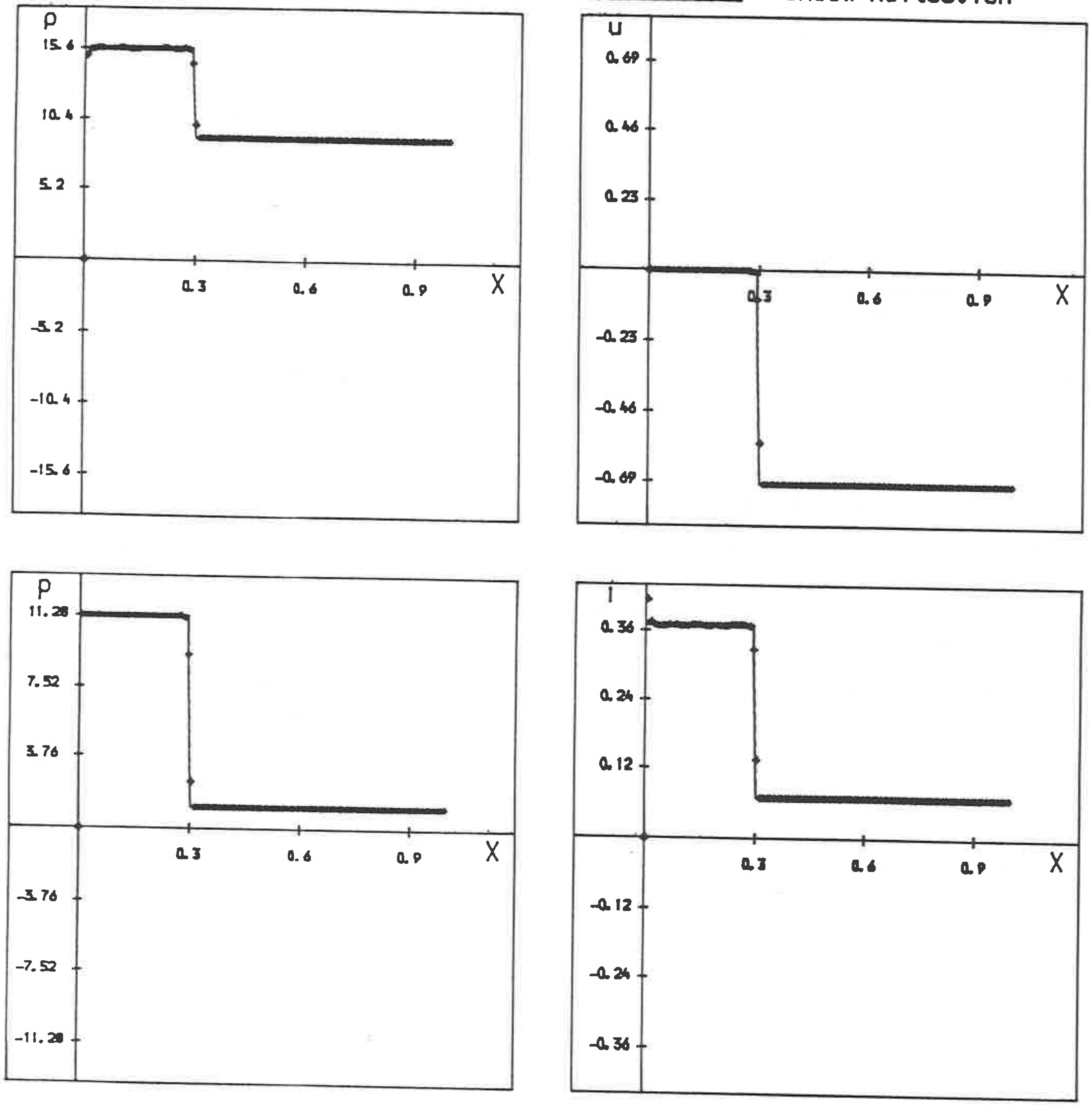
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.351

Figure 16

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state,
for Copper due to R.K. Osborne
100 Mesh points
117 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0027$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 8.900
u = -0.700
p = 1.128
(I = 0.069)

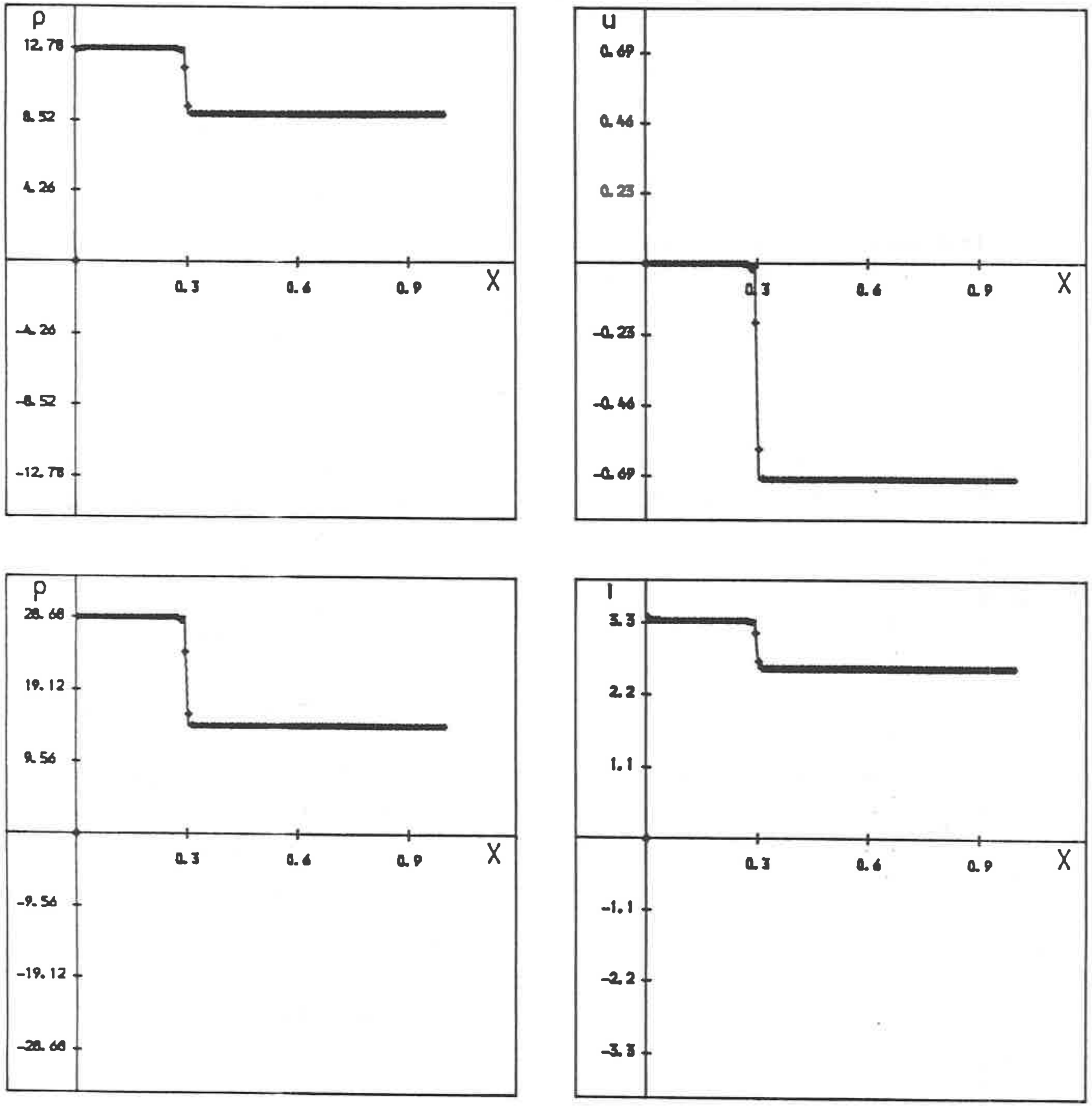
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.321

Figure 17

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state :
for Copper due to R.K.Osborne
100 Mesh points
113 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0016$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 8.900
u = -0.700
p = 14.351
(I = 2.594)

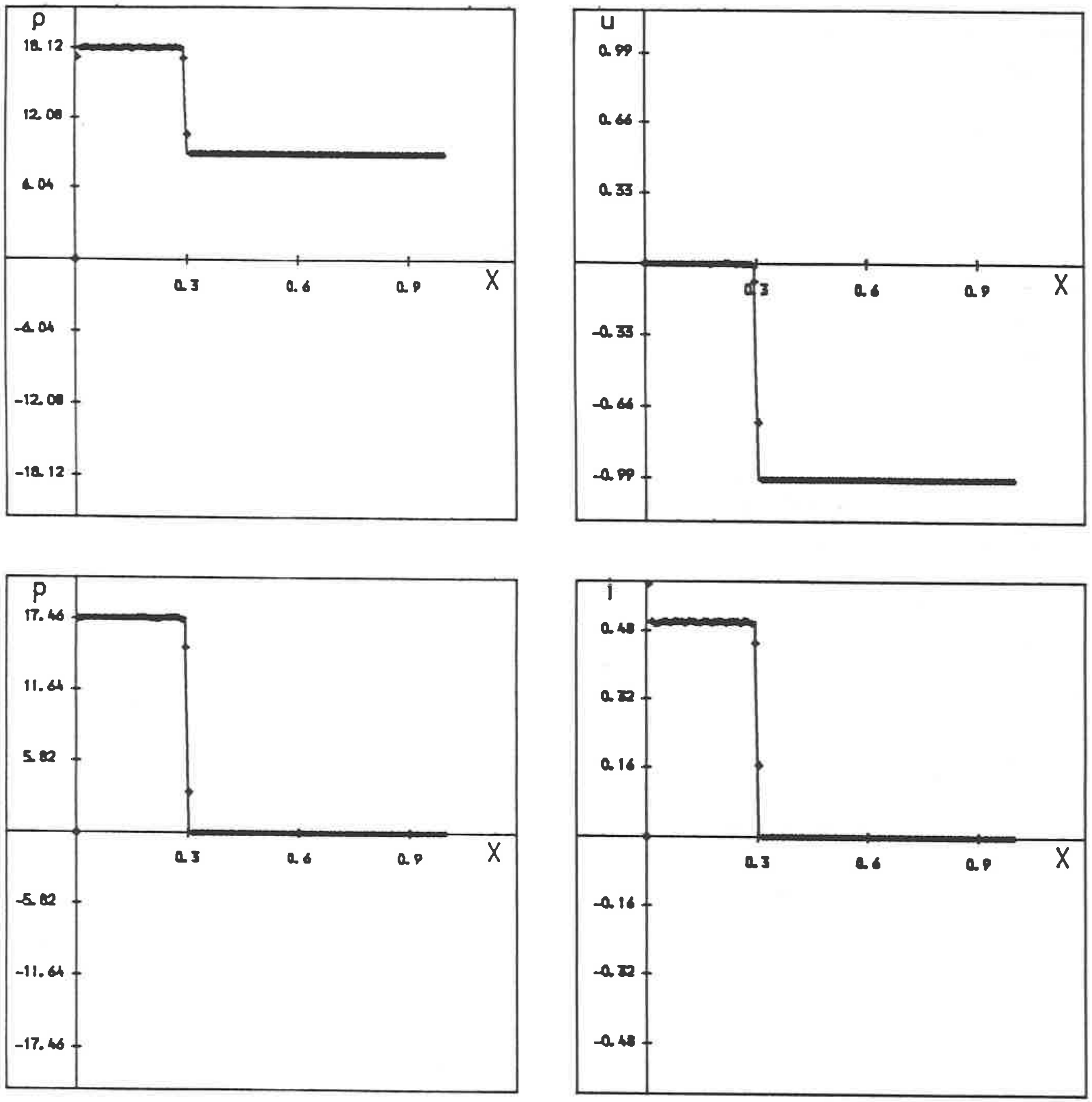
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.186

Figure 18

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
i - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

General equation of state :
for Copper due to R.K. Osborne
100 Mesh points
128 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0024$
Pressure ratio = ∞
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 8.900
u = -1.000
p = 0.000
(i = 0.000)

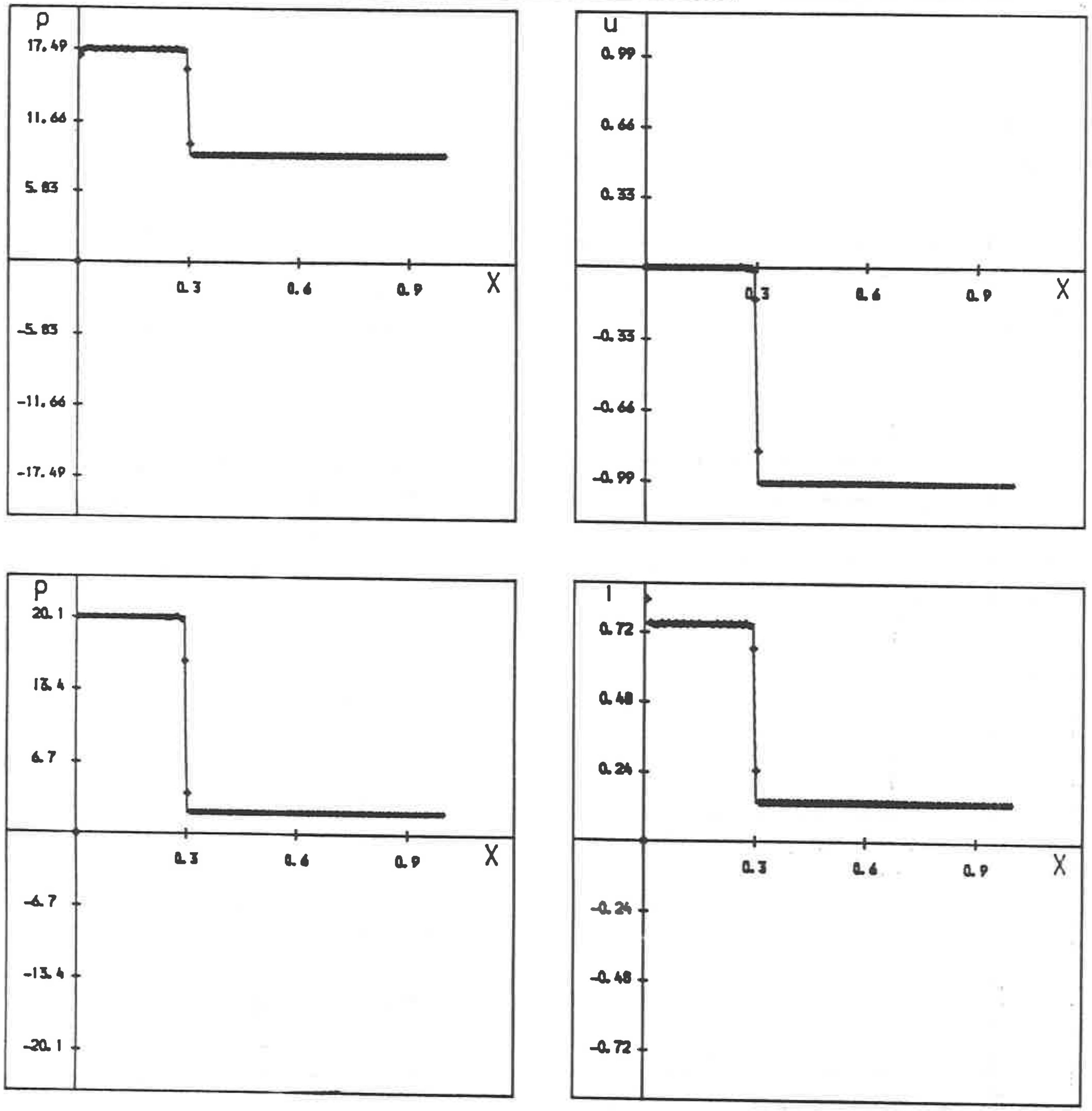
0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.311

Figure 19

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◇◇◇◇◇◇ Approximate solution

PARAMETERS

General equation of state ,
for Copper due to R.K. Osborne
100 Mesh points
135 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0021$
Pressure ratio = 10
'Superbee' limiter used

INITIAL CONDITIONS

$\rho = 8.900$
$u = -1.000$
$p = 2.013$
$(I = 0.137)$

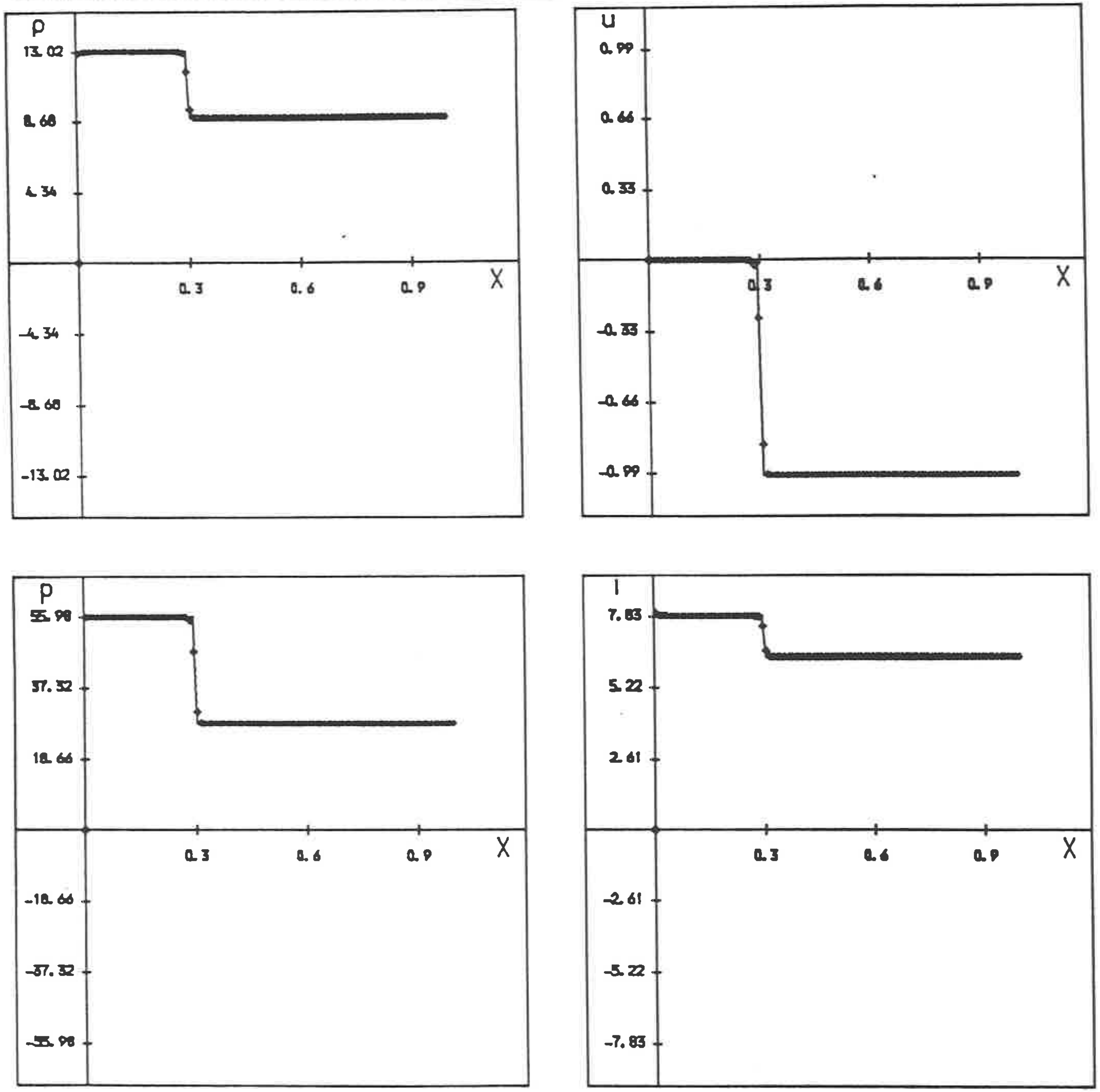
0 1

Reflected Boundary Conditions
at $x = 0$

at time t = 0.288

Figure 20

SOLUTION OF THE EULER EQUATIONS WITH SLAB SYMMETRY - Shock Reflection



KEY

ρ - Density
u - Velocity
p - Pressure
I - Internal energy

—— Exact solution
◊◊◊◊◊◊ Approximate solution

PARAMETERS

General equation of state :
for Copper due to R.K. Osborne
100 Mesh points
117 Time steps
$\Delta x = 0.01$
$\Delta t = 0.0012$
Pressure ratio = 2
'Superbee' limiter used

INITIAL CONDITIONS

ρ = 8.900
u = -1.000
p = 27.996
(I = 6.347)

0 1

Reflected Boundary Conditions
at x = 0

at time t = 0.139

Figure 21

ACKNOWLEDGEMENTS

I would like to express my thanks to Dr. M.J. Baines for useful discussions and to D.L. Youngs for suggesting the test problem given in §5.

I acknowledge the financial support of A.W.R.E., Aldermaston.

7. CONCLUSIONS

We have extended the one dimensional version of Roe's scheme to include a general equation of state, and we have achieved satisfactory results for the problem of shock reflection. In addition, we have seen that the algorithm is computationally efficient.

In the future we hope to extend our scheme to three dimensions using operator splitting.

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