Source Terms

and

Conservation Laws:

A Preliminary Discussion

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Abstract

The effect of source terms on basic philosophies of algorithm design for hyperbolic conservation laws are investigated. Two particular philosophies are studied – pointwise data which leads to interpolation-advection schemes and cell averaged data which produces the well known Godunov type scheme.

This report does not attempt to provide answers, rather to provide food for thought.

1. Introduction

Much effort has been devoted to the design of accurate and robust numerical methods for the solution of hyperbolic conservation laws

$$u_t + f(u)_x = 0 \tag{1.1}$$

and many of the techniques developed are now well established and in common use, at least within the research community. However for physical processes the modelling equations are often inhomogeneous with terms on the right hand side, due to friction for example.

$$u_t + f(u)_x = b(u) \tag{1.2}$$

To date the treatment of such source terms has been reasonably ad hoc - especially where high resolution TVD schemes are involved, since these schemes seek to prevent an increase in variation of the solution although such an increase may be precisely what the source term physically does.

It is not just this interaction, however, which is at issue. The fundamental question of how to treat the souce term still remains. Should it be pointwise?

i.e. for (1.2) should we simply add, for example, the term

$$\Delta t \, b(u_k^n) \tag{1.3}$$

to our scheme? It is certainly an approximation but takes no account of

the characteristics of the equation. Another alternative would be to calculate an average value

$$\Delta t \; \bar{b} \qquad\qquad (1.4)$$

in a computational cell and then upwind it. This far more physical in its approach.

This report does not purport to settle the treatment of source terms, but rather to take a brief look, at a very basic level, at the effect source terms have on basic philosophies used in developing schemes for (1.1). The reader is also referred to other literature, such as [14],[15],[16],[17] for a sampling of treatments applied so far.

In section 2 two basic viewpoints of the data and the subsequent construction of schemes are recounted. Then in section 3 the effect of source terms on the pointwise data interpretation is investigated, followed in section 4 by the effect on cell average data. In section 5 issues common to both viewpoints are highlighted.

2. The Homogenous Equation - Two Different Viewpoints

We consider first the scalar homogenous conservation law

$$u_t + f(u)_x = 0 \qquad\qquad (2.1)$$

and in particular, as is usual when developing schemes for such equations, we concentrate on the linear advection equation

$$u_t + au_x = 0 \qquad\qquad (2.2)$$

where a is a constant, usually assumed positive, as is the case here.

In designing schemes for equation (2.2) various assumptions and interpretations may be taken, many of which are equivalent for this special linear case. For the purposes of this work we concentrate on two particular viewpoints, firstly where the data is interpreted as point values of the underlying function which is represented by some piecewise polynomial interpolant before being advected according to (2.2) and sampled at the nodal positions. The alternative viewpoint we consider is where the data is interpreted as representing cell averages of the underlying function to which (2.1) is applied and solved, an averaging process producing the updated solution.

The latter interpretation gives rise to what are known as Godunov methods, named after the pioneering scheme of this form [1]. The former interpretaiton has been dubbed interpolation-advection schemes and it is with these that we start our analysis.

2.1 Interpolation-Advection Schemes

It is well known [2] that classical constant coefficient schemes for the solution of (2.2) may be interpreted as advecting a piecewise polynomial interpolant of point values of $u(x,t)$ and sampling at the

nodal positions to obtain an updated solution.

For example, consider a piecewise linear interpolation of nodal
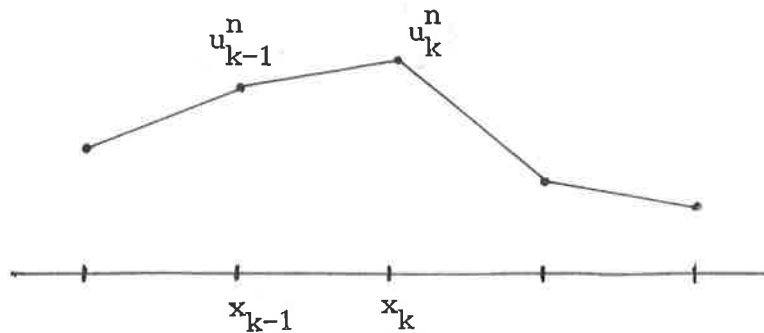
values at time t as depicted in Figure 2.1.

Figure 2.1   A Piecewise Linear Interpolation of Nodal Values

In the cell (x_{k-1}, x_k) this interpolant can be written as

$$\tilde{u}_{k-\frac{1}{2}}(x) = u_{k-1}^n + (x-x_{k-1}) \frac{(u_k^n - u_{k-1}^n)}{x_k - x_{k-1}} \qquad x \in (x_{k-1}, x_k) \qquad (2.3)$$

with similar expressions holding in other cells.

If we apply equation (2.2) to this piecewise linear interpolant the

effect is to shift the whole interpolant a distance $a\Delta t$ to the right

in a time Δt without any distortion of its form, since u is

constant on its characteristics which are parallel straight lines,

dx/dt = a . This process is depicted in Figure 2.2.

We can now obtain updated values $u(x_i, t+\Delta t)$ by sampling the

advected interpolant $\bigcup_k \tilde{u}_{k-\frac{1}{2}}$ at the nodal positions as indicated by the

square symbols in Figure 2.2.
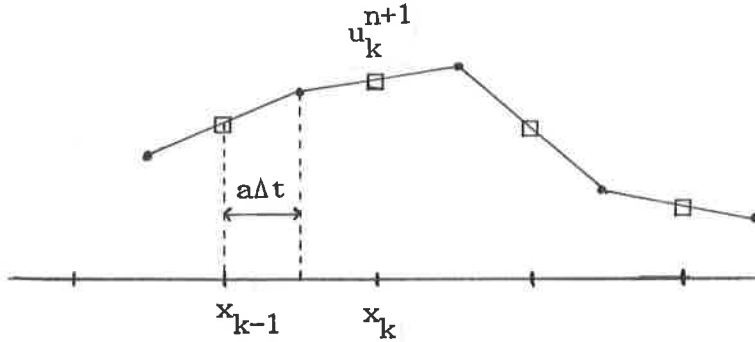
<u>Figure 2.2   Advection of the Interpolant</u>

If we assume that the distance $a\Delta t$ moved by the interpolant is less than one cell's width Δx , then the updated value is obtained by evaluating (2.3) at $x_k - a\Delta t$, i.e.

$$u_k^{n+1} = \tilde{u}(x_k - a\Delta t)$$

$$= u_{k-1}^n + (x_k - a\Delta t - x_{k-1}) \frac{(u_k^n - u_{k-1}^n)}{x_k - x_{k-1}}$$

$$= u_{k-1}^n + \frac{(\Delta x - a\Delta t)}{\Delta x} (u_k^n - u_{k-1}^n)$$

$$= u_k^n - \frac{a\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) \qquad\qquad (2.4)$$

which is instantly recognisable as the Cole-Murman first order upwind scheme [3] for (2.2).

Instead of the linear interpolation (2.3) we could instead have

used a quadratic interpolant, based say on the knots x_{k-1}, x_k, x_{k+1},
for the interval $x \in (x_{k-1}, x_k)$. The interpolant is then given by

$$\tilde{u}_{k-\frac{1}{2}}(x) = \frac{(x-x_k)(x-x_{k+1})}{2\Delta x^2} u_{k-1}^n - \frac{(x-x_{k-1})(x-x_{k+1})}{\Delta x^2} u_k^n$$

$$+ \frac{(x-x_{k-1})(x-x_k)}{2\Delta x^2} u_{k+1}^n \qquad x \in (x_{k-1}, x_k) \qquad (2.5)$$

as shown in Figure 2.3 and the advection, sampling illustrated in
Figure 2.4.

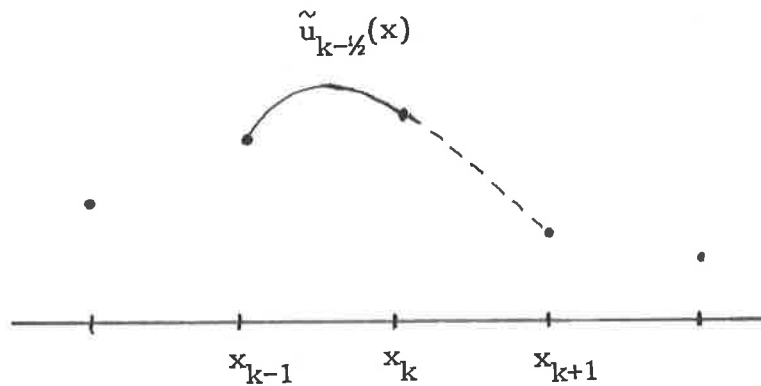Figure 2.3   Piecewise Quadratic Interpolant

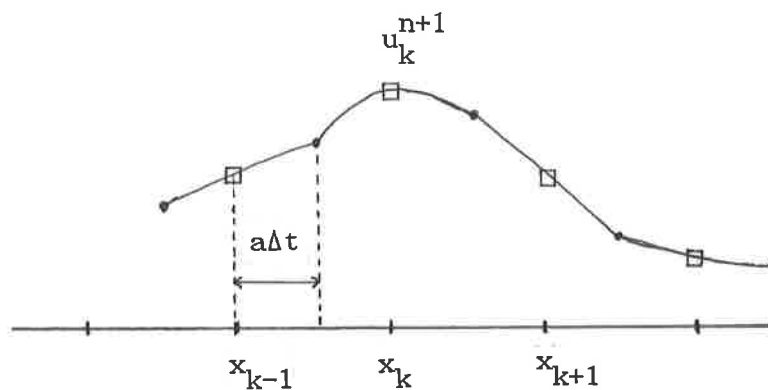Figure 2.4   Advection and Sampling of Interpolant

Algebraically the sampling gives the updated value as

$$u_k^{n+1} = \tilde{u}_{k-\frac{1}{2}} (x_k - a\Delta t)$$

$$= \frac{(-a\Delta t) (-\Delta x - a\Delta t)}{2\Delta x^2} u_{k-1}^n - \frac{(\Delta x - a\Delta t) (-\Delta x - a\Delta t)}{\Delta x^2} u_k^n$$

$$+ \frac{(\Delta x - a\Delta t) (a\Delta t)}{2\Delta x^2} u_{k+1}^n$$

$$= \quad \ldots\ldots\ldots\ldots\ldots$$

$$= u_k^n - a\frac{\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) - \frac{a\Delta t}{2\Delta x} (1 - a\frac{\Delta t}{\Delta x}) (u_{k+1}^n - 2u_k^n + u_{k-1}^n)$$

$$(2.6)$$

which is the well-known second order accurate Lax-Wendroff scheme [4].

Using this interpretation of the scheme it is evident from Figure 2.4 why the Lax-Wendroff scheme produces spurious oscillations behind discontinuities, since any rapid change in adjacent data values will give an interpolant which is not bounded by its values at the knots, and so, on sampling the advected data representation, updated values of u_k^{n+1} may be obtained which violate the local maximum principle associated with (2.2) [5].

If instead of using knots x_{k-1}, x_k, x_{k+1} for the quadratic interpolant in (x_{k-1}, x_k) we had used x_{k-2}, x_{k-1}, x_k then the outcome of the interpolation-advection process would have been to produce the second order upwind scheme of Warming & Beam [6] which will also produce oscillations in a similar manner to Lax-Wendroff; however for this scheme they appear forward of any discontinuity owing to the

shifted interpolation stencil.

Although not the subject of this report, we disgress here
momentarily to remark that Flux Limiter schemes [7] applied to (2.2) may
be interpreted within this framework by regarding them as specifying a
value $\hat{u}$ to replace the interpolation knot at x_{k+1} in (2.5). This
value is chosen so that the quadratic interpolant through
(x_{k-1}, u_{k-1}) , (x_k, u_k) , $(x_{k+1}, \hat{u})$ has no internal extrema in the
interval (x_{k-1}, x_k) . A typical situation is shown in Figure 2.5, the
absence of any extrema in the interpolant ensuring the well-known TVD
property [8] of the scheme obtained by advection and sampling the
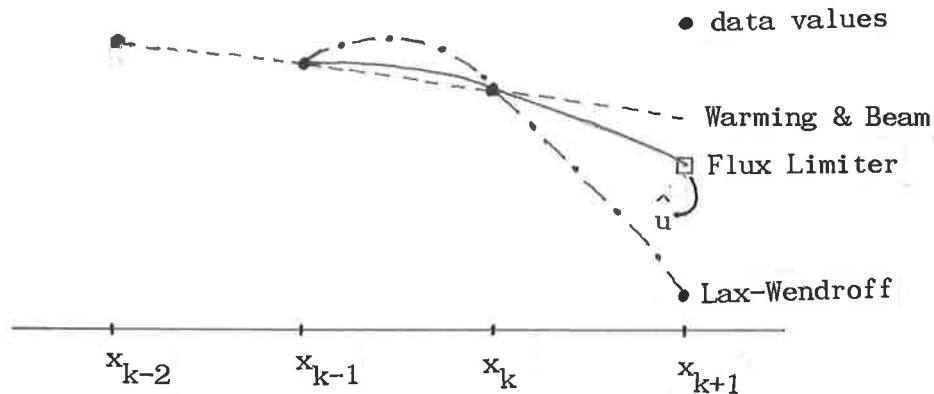interpolant.

Figure 2.5   Limiter Interpolation

Schemes of the interpolant-advection form for the linear advection
equation (2.2) are usually extended to the more general nonlinear
equation (2.1) by replacing the constant wave speed by the approximation
$(f_k - f_{k-1})/(u_k - u_{k-1})$ to $f'(u)$ in each cell which serves not only to
introduce the nonlinear wave speed into the scheme but also to maintain

the conservation form of the scheme which is all-important for shock capturing schemes.

[A full discussion of conservation form may be found in [4]; however, to summarise here, a scheme is said to be written in conservation form if it is expressed as

$$u_k^{n+1} = u_k^n - \lambda(h_{k+\frac{1}{2}} - h_{k-\frac{1}{2}}) \qquad (2.7)$$

where $\quad \lambda = \frac{\Delta t}{\Delta x}$, the usual mesh ratio

and $\quad h_{k-\frac{1}{2}} = h_{k-\frac{1}{2}}(u_{k-1}, \ldots, u_{k+r})$

is a consistent numerical flux function such that

$$h(u, \ldots, u) = f(u) \ .$$

The importance of conservation form is summarized by the Lax-Wendroff theorem which says that if the scheme convex then the solution is a weak solution of (2.1); this in turn implies that shock locations and speeds will be correct; hence the term Shock Capturing Schemes].

We shall now look at a particular class of schemes for (2.1), commonly known as Godunov Schemes; in particular we shall concentrate on the definitive scheme of the class. As we will see, such schemes take a somewhat different interpretation of the data.

2.2 Godunov Schemes

Unlike the point value interpretation given to the data u_k^n by the

interpolation-advection approach. Godunov schemes regard the data as representing cell averages of the underlying function $u(x,t)$, i.e.

$$u_k^n = \frac{1}{\Delta x} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} u(x,n\Delta t)dx \ . \tag{2.8}$$

Given a set of cell averages then the simplest way to envisage the underlying function $u(x,t)$ is as a set of piecewise constant steps as shown in Figure 2.6. By (2.8) the area under each step must be equal to $\Delta x u_k^n$ - for this type of scheme it is this feature which gives rise to the conservation property.
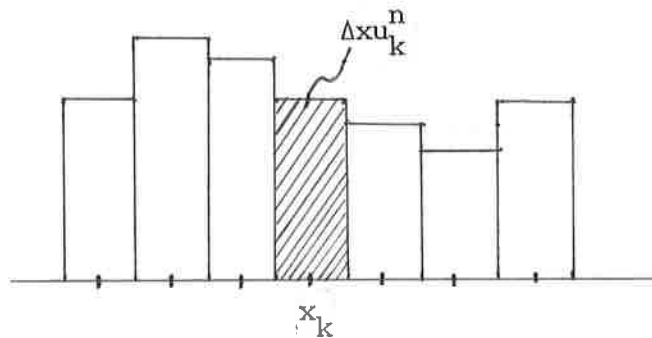

Figure 2.6 Piecewise constant data representation

Having settled on this data representaiton Godunov's scheme next treats each interface between cells as a Riemann problem and solves this

exactly. Hence for the interface at $x_{k-\frac{1}{2}}$ we solve

$$v_t + f(v)_x = 0$$

(2.9)

with $\qquad v(x,t) = \begin{cases} u_{k-1}^n & x < x_{k-\frac{1}{2}} \\ u_k^n & x > x_{k-\frac{1}{2}} \end{cases}$

which defines a classic Riemann Problem – i.e. solving the conservation

law for a single jump in otherwise constant data.

For a scalar conservation law the solution to (2.9) is either

(i) a discontinuity propagating with speed

$$s = \frac{f(u_k^n) - f(u_{k-1}^n)}{u_k^n - u_{k-1}^n}$$

(2.10)

according to the well-known jump condition (see e.g. [9] or [10]),

giving

$$v(x, t+\tau) = \begin{cases} u_{k-1}^n & x < x_{k-\frac{1}{2}} + s\tau \\ u_k^n & x > x_{k-\frac{1}{2}} + s\tau \end{cases}$$

(2.11)

or

(ii) an expansion wave given by

$$v(x, t+\tau) = \begin{cases} u_{k-1}^n & x < f'(u_{k-1}^n)\tau \\ u_{k-1}^n + \dfrac{(u_k^n - u_{k-1}^n)}{f_k' - f_{k-1}'} (x/\tau - f'_{k-1}) & \tau f'(u_{k-1}^n) < x < f'(u_k^n)\tau \\ u_k^n & x > f'(u_k^n)\tau \end{cases}$$

(2.12)

The appropriate solution is selected by the Entropy Condition (see e.g. [9] or [10]), here

case (i) if $f'(u_{k-1}^n) \geq s \geq f'(u_k^n)$.

case (ii) otherwise.

Having calculated the exact solution $v(x, t+\tau)$ the updated cell averages are calculated:

$$u_k^{n+1} = \frac{1}{\Delta x} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} v(x, t+\Delta t) \, dx \, . \tag{2.13}$$

Using the conservation property of (2.8) we can combine the solution and averaging stages. We illustrate this here for the case $0 \leq f' \leq \frac{\Delta x}{\Delta t}$. (The second inequality is necessary for the solution of neighbouring problems not to interact).

In the (x, t) plane we have one of the situations shown in Figure 2.7, for this case both may be treated in the same manner.
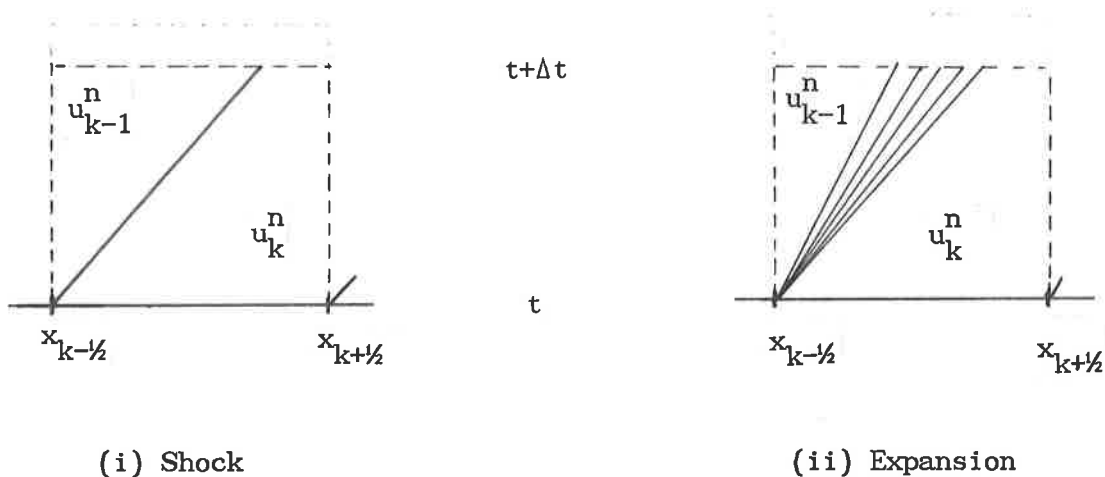

(i) Shock   (ii) Expansion

Figure 2.7   Riemann Solutions

If we integrate (2.9) around the box $(x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}) \times (t, t+\Delta t)$ we obtain

$$\int_{t}^{t+\Delta t} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} v_t + f(v)_x \; dxdt = 0$$

therefore

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \{v(x, t+\Delta t) - v(x, t)\}dx + \int_{t}^{t+\Delta t} \{f(v(x_{k+\frac{1}{2}}, t)) - f(v(x_{k-\frac{1}{2}}, t))\}dt = 0$$

therefore

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} v(x, t+\Delta t)dx = \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} v(x, t)dx - \int_{t}^{t+\Delta t} \{f(v(x_{k+\frac{1}{2}}, t)) - f(v(x_{k-\frac{1}{2}}, t))\}dt$$

i.e.

$$u_k^{n+1} = u_k^n - \frac{\Delta t}{\Delta x} (f_k^n - f_{k-1}^n) \tag{2.14}$$

on observing $f(v(x_{k+\frac{1}{2}}, t)) = f(u_k^n)$ for this case.

Similar integrations yield expressions for the scheme for the other cases $f' < 0$, $f'_k < 0 < f'_{k-1}$ and $f'_{k-1} < 0 < f'_k$.

This process leads to a first order accurate scheme. If, however, instead of piecewise constant the underlying function is envisaged as piecewise linear, as in Figure 2.8 for example, a similar process can lead to a second order accurate scheme. Now at cell interfaces a generalised Riemann problem must be solved for the jump in the linear representation, in practice this is usually done approximately and this
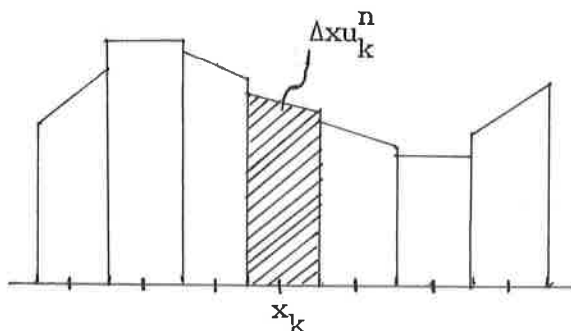
approximate solution reaveraged.

Figure 2.8   Piecewise Linear Representation

For conservation we must have that the area under each linear segment is equal to $\Delta x u_k^n$ by (2.8); however we are free to choose the slopes. An arbitrary choice may well lead to spurious oscillations in the solution, and so care must be taken not to increase the total variation of the representation - van Leer's MUSCL scheme [11] illustrates how this may be achieved.

By using higher order representations higher order accuracy may be obtained, such as in Woodward and Colella's PPM [12], we end this chapter, however, by mentioning a class of schemes which is in some sense a cross-breed of the interpolation-advection and the Godunov viewpoints. These are the ENO schemes of Osher, Chakraworthy, Harrten and Engquist [13]. We shall not pursue them further in this report but they should be mentioned briefly for the sake of completeness.

2.3 ENO Schemes

ENO schemes again view the data points as being cell averages and uses them to construct an accumulative average

$$w(x) = \int u(y) \, dy \qquad (2.15)$$

whence $\qquad w(x_{k+\frac{1}{2}}) = \sum^{k} \Delta x \, u_i^n \, , \qquad (2.16)$

from (2.8).

This is then interpolated by high-order continuous piecewise polynomials, in such a way as to prevent oscillations (see [13]), and this interpolant differentiated to obtain a piecewise continuous polynomial interpolant to the underlying function u . This interpolant will have discontinuities at cell interfaces, $x_{k+\frac{1}{2}}$, where an approximate generalised Riemann solution can be made before reaveraging. The reader is referred to the original papers for full details.

In the next section we investigate possible ways of incorporating a source term in the interpolation-advection framework whilst in section 4 we return to Godunov type schemes and investigate the inclusion of source terms in such schemes.

3. The Non-homogeneous Equation and Interpolation-Advection

In this and the next section we consider the non-homogeneous equations

$$u_t + f(u)_x = b(u) \tag{3.1}$$

and

$$u_t + au_x = b(u) \ . \tag{3.2}$$

For simplicity we have confined ourselves to the case where the non-homogeneous, or source, term is a function of u only.

If we look at these equations analytically the first point to note is that, whereas in the homogeneous case u is constant along the characteristics of the equations, this is no longer the situation.

Indeed, simple calculus gives us that

$$\frac{du}{dt} = b(u) \tag{3.3}$$

along the characteristics

$$\frac{dx}{dt} = f'(u) \quad \text{(or} \quad a \quad \text{as in the case of (3.2)).} \tag{3.4}$$

One immediate implication is that the characteristics need no longer be straight lines. Certainly for (3.2) we still have straight, parallel characteristics due to the linearity of the equation, however for (3.1) since the characteristic slope is given as a function of u , this will no longer be constant for each characteristic.

In this section we are concerned mainly with (3.2) and so we will defer the nonlinear aspects of (3.3) until we return to Godunov schemes. For the moment however, we look again at the interpolation-advection viewpoint.

3.1 Interpolation-Advection

The interpolation stage is entirely unaffected by the presence of the source term b(u) in (3.2). The data points are connected by segments of polynomials as before. However, due to (3.3), the interpolant is no longer advected without change of form and so we must take care when sampling to obtain the updated data point.

Consider the piecewise linear interpolant, depicted in Figure 2.1 and given, in each cell, by (2.3). Since the characterisics of (3.2) are still straight and parallel, we can trace back the characteristic passing through $(x_k, t+\Delta t)$ to where it intersects the interpolant at $(\hat{x}, t)$ – see Figure 3.1.
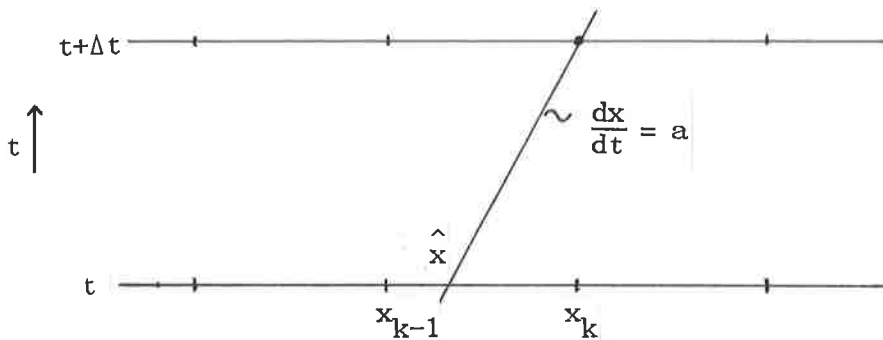

Figure 3.1 Tracing back the characteristic

It is easy to see that

$$\hat{x} = x_k - a\Delta t \ .$$

(3.5)

For the homogeneous case we would now simply substitute $\hat{x}$ into (2.3) to obtain u_k^{n+1} since for that case u is constant along the characteristic. Here however we must integrate (3.3) along the characteristic to obtain our updated value, i.e. solve

$$
\left.\begin{array}{l}
\dfrac{dv}{d\tau} = b(v) \\[2ex]
v(0) = \tilde{u}_{k-\frac{1}{2}}(\hat{x})
\end{array}\right\} \tag{3.6}
$$

where $\tilde{u}_{k-\frac{1}{2}}$ is given by (2.3), $v(\tau)$ is the restriction of u to the characteristic, i.e.

$$
v(\tau) = u(x_k - a(\Delta t - \tau), \ t + \tau) \tag{3.7}
$$

and the updated value is given by

$$
u_k^{n+1} = v(\Delta t) \ . \tag{3.8}
$$

Note that by the introduction of v in terms of u and __not__ $\tilde{u}$ we have __not__ made the (unreasonable) assumption that the interpolant remains linear during the advection.

We now write

$$
\begin{aligned}
v(\Delta t) &= v(0) + \int_0^{\Delta t} b(v) \ d\tau \\[2ex]
&= \tilde{u}_{k-\frac{1}{2}}(x_k - a\Delta t) + \int_0^{\Delta t} b(v) \ d\tau \ ,
\end{aligned} \tag{3.9}
$$

then by the analysis of section 2.1, equation (2.4), we obtain

$$u_k^{n+1} = u_k^n - a\frac{\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \int_0^{\Delta t} b(v) \, d\tau \qquad (3.10)$$

where an ODE scheme or quadrature rule may be used to evaluate the last term to some given accuracy via (3.6) and (3.9). For example, using the Euler scheme gives

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \Delta t \, b \, (\tilde{u}_{k-\frac{1}{2}}(x_k - a\Delta t))$$

$$= u_k^n - \frac{a\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \Delta t \, b \left[u_k^n - \frac{a\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) \right] . \qquad (3.11)$$

We can further approximate by expanding the last term, assuming $b'(u)$ is small (i.e. b approximately linear) producing

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \Delta t \, b(u_k^n) - \Delta t \, \frac{a\Delta t}{\Delta x} (b(u_k^n) - b(u_{k-1}^n))$$

$$(3.12)$$

This is the same result we would have obtained if we had assumed the interpolant advected as a straight line in each cell.

Note, since $\Delta t/\Delta x$ is a constant the last two terms of (3.12) are of the same order in Δt .

3.2 Higher Order Interpolants

All the analysis of the previous section up to and including (3.9), holds for higher order interpolants, including any TVD devices such as Flux Limiters applied to them. We note however that (3.2) does not

imply any such TVD property for its analytic solution (indeed many choices of b(u) will lead to a solution which should clearly violate such a property) and so care must be taken to distinguish between spurious and inherent variation increases.

Application of a limited interpolant as illustrated in Figure 2.5 will remove spurious increases in variation brought about by the interpolant whilst allowing the integration of (3.6) to introduce any inherent growth. Viewing (3.9), where we now take $\tilde{u}$ to be a limited interpolant then shows an essential decoupling of TVD devices and treatment of source terms. The only effect the limited interpolant has on the souce treatment is in the initial condition of (3.6) and <u>not</u> on the technique for its integration.

On the otherhand, we might notice that we can rewrite (3.12) in terms of $u + b(u)\Delta t$ and be tempted to apply higher order TVD schemes to this quantity. Indeed this is similar to what has been advocated for some situations (e.g. [14],[15]). However in obtaining (3.12) the assumption that b(u) was near linear in u was made and so generally we would not be cetain that inherent growth was not being stunted due to this approximation which essentially effects the treatment of the source term and not the interpolation of data.

These heuristic arguments tie in with observations made in [16].

In the following section we investigate the effect of the source term on the Godunov approach to solving conservation laws.

4. Godunov Type Methods with Source Terms

We now turn our attention back to Godunov type methods where the data is considered as cell averages with Riemann solutions at cell interfaces being reaveraged to obtain updated values.

Inherent in the solution of Riemann problems is the jump condition (2.10) and so we must investigate its validity when source terms are present.

Although we omit the details here, if we follow Smoller's proof [9] of the jump condition, it is easily seen that the presence of a non-homogeneous term cancels during the course of the proof verifying that the jump condition (2.10) holds for both the homogeneous and inhomogeneous case.

Even with the validity of the jump condition guaranteed, however, we must be careful with its application. We should remember that it gives an _instantaneous_ shock speed. Whereas a discontinuity between constant states propagates with constant speed in the homogeneous case this will not happen in the inhomogeneous case since generally the value of u on either side of the discontinuity will no longer remain constant due to (3.3). The exception to this of course is the linear equation (3.2) whose discontinuities (not true shocks) always propagate with speed a.

We shall concentrate here on Godunov's original piecewise constant representation as depicted in Figure 2.6, the data representation again being unaffected by the presence of the source term. However, unlike in section 2.2 we do not start here with the nonlinear equation since it is informative to study first the linear equation (3.2).

4.1 The Linear Equation

Assuming, as usual, that the constant a in (3.2) is positive the situation in the (x,t) plane is as shown in Figure 4.1.
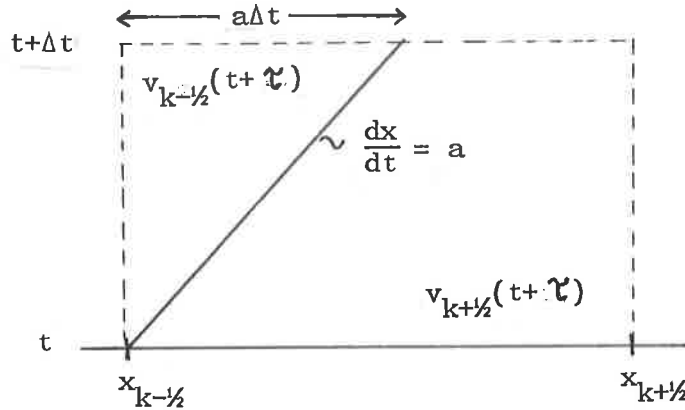

Figure 4.1   The Linear Equation Problem

Note that the Riemann solution v is no longer constant on either side of the discontinuity but is now a function of time, since we are solving

$$v_t + av_x = b(v)$$

(4.1)

$$v(x,t) = \begin{cases} u^n_{k-1} & x < x_{k-\frac{1}{2}} \\ u^n_k & x > x_{k-\frac{1}{2}} \end{cases} .$$

Using either the technique of integrating (4.1) over $(x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}) \times (t, t+\Delta t)$, or, as is easily done for this case, by direct

averaging gives

$$u_k^{n+1} = \frac{1}{\Delta x} \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} v(x, t+\Delta t) \, dx$$

$$= v_{k+\frac{1}{2}} (t+\Delta t) - a \frac{\Delta t}{\Delta x}(v_{k+\frac{1}{2}}(t+\Delta t) - v_{k-\frac{1}{2}}(t+\Delta t)) \qquad (4.2)$$

$$= u_k^n - a \frac{\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \int_0^{\Delta t} b(v_{k+\frac{1}{2}}(t+\tau)) \, d\tau$$

$$- a \frac{\Delta t}{\Delta x} \int_0^{\Delta t} [b(v_{k+\frac{1}{2}}(t+\tau)) - b(v_{k-\frac{1}{2}}(t+\tau))] d\tau \; .$$

$$(4.3)$$

If we now use the Euler approximation we obtain

$$u_k^n - a \frac{\Delta t}{\Delta x} (u_k^n - u_{k-1}^n) + \Delta t \, b(u_k^n) - a \frac{\Delta t}{\Delta x} (b(u_k^n) - b(u_{k-1}^n))$$

which is precisely (3.12). Note that this time linearity of b has not explicitly been assumed.

We now turn to the nonlinear problem (3.1), the Riemann solution for this equation being far from simple.

4.2 The Non-Linear Equation

For the non-linear equation (3.1) the Riemann problem which must be

solved is

$$v_\tau + f(v)_x = b(v)$$

$$v(x,0) = \begin{cases} u_{k-1}^n & x < x_{k-\frac{1}{2}} \\ u_k^n & x > x_{k+\frac{1}{2}} \end{cases} \tag{4.4}$$

where we have shifted the time origin to time t for notational convenience and adopted a new time variable τ running from 0 to Δt.

The non-homogeneity of the problem has a number of effects on the Riemann solution. As has already been mentioned the shock speed will no longer be constant, leading to a situation depicted in Figure 4.2, where it is clear that the approximation of
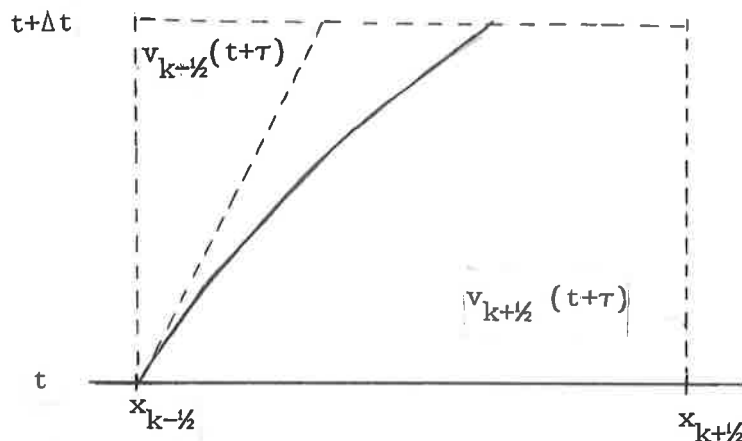

Figure 4.2   A Curved Shock

a constant shock speed calculated at time t (dotted line) may not be satisfactory.

We can obtain an indication of how well a constant speed will approximate the true shock by differentiating the jump condition (2.10), here suffixes L and R denoting values to the left and right of the discontinuity.

$$\dot{s} = \frac{f'_R \dot{v}_R - f'_L \dot{v}_L}{v_R - v_L} \;-\; s \frac{\dot{v}_R - \dot{v}_L}{v_R - v_L}$$

$$= \frac{(f'_R - s)\dot{v}_R - (f'_L - s)\dot{v}_L}{v_R - v_L}$$

$$= \frac{(f'_R - s)b_R - (f'_L - s)b_L}{v_R - v_L} \; . \tag{4.5}$$

For example, if $f(v) = \tfrac{1}{2}v^2$ — the inviscid Burger's equation, we have

$$\dot{s} = \tfrac{1}{2}(b_R + b_L) \tag{4.6}$$

which need not be small, and so a small Δt would be needed for a constant speed to be a reasonable approximation.

Another, less obvious, consequence of the source term is that since the characteristics themselves are not straight lines and u not constant on them it would be possible for a solution starting as an expansion wave, say, to shock during the course of the solution. Another possibility is a shock dying out to produce a uniform state throughout the cell.

It must, by now be obvious that approximations will have to be made during the solution. Indeed, apart from the difficulty of solution of

the Riemann problem, the averaging of the solution may not be a trivial exercise.

The most drastic permissible approximation probably is to model the solution using only shocks with constant speed, the resulting analysis being as in section 4.1 with the constant a being replaced cellwise by the initial shock speed s given by (2.10). As noted above small timesteps may be required to obtain reasonable results with such an approximation.

In theory higher order data representation such as MUSCL's piecewise linear segments could be used, however the complexities introduced by the source term would make the problem even more intractable without gross approximations/small timesteps. Consequently this aspect is not considered further here.

In the next section we note some other aspects introduced by source terms which do not owe allegiance to either viewpoint recounted in this report.

5. Other Issues: Shock Speed, Stiffness and Chaos

We conclude this report by briefly mentioning other consequences of source terms which arise independently of the view point being taken.

Firstly, inaccurate shock speeds may arise as a consequence of grid resolution and/or inaccurate integration of (3.3). Take, for instance, the equation

$$u_t + au_x = \alpha u(1-u) \tag{5.1}$$

with step initial data

$$u_o(x) = \begin{cases} 1 & x < \tfrac{1}{2} \\ 0 & x > \tfrac{1}{2} \end{cases} . \tag{5.2}$$

Clearly, if a scheme used to solve this combination introduced no points in the discontinuity (as should be the case) (5.1) reduces to the linear homogeneous advection equation and the step moves righwards with speed a .

However, in general this will not happen, intermediate values of u being introduced signifying the position of the discontinuity within a cell. What happens now? Consider, instead of (5.2), initial data

$$u_o(x) = \begin{cases} 1 & x < \tfrac{1}{2} \\ \tfrac{1}{2} & x = \tfrac{1}{2} \\ 0 & x > \tfrac{1}{2} \end{cases} . \tag{5.3}$$

Along the characteristics

$$\frac{du}{dt} = \alpha u(1-u) \tag{5.4}$$

which has solution

$$u(t) = \frac{Ae^{\alpha t}}{1+Ae^{\alpha t}} \qquad (5.5)$$

on a particular characteristic. In particular, on the characteristic passing through $(\frac{1}{2},0)$ we have $A = 1$, i.e.

$$u(t) = \frac{e^{\alpha t}}{1+e^{\alpha t}} \qquad (5.6)$$

hence as $\alpha t \to \infty$, $u \to 1$. To be more precise, u gets within ϵ of 1 , i.e. $u = 1 - \epsilon$, when

$$\alpha t = \ln \left[\frac{1-\epsilon}{\epsilon}\right] \qquad (5.7)$$

and so, for example, $u \simeq .99$ when $\alpha t \simeq 4.6$. To graphical accuracy therefore, replacing (5.2) by (5.3) will cause the discontinuity to have advanced one extra cell by the time $\alpha t = 4.6$ - i.e. the discontinuity moves too fast for the solution of (5.1) and (5.2) if the scheme introduces values between 0 and 1 in the discontinuity. This has been assuming the ODE solver for (5.4) is exact, which will not be the case and so the discrepancy may be even more pronounced.

Other source terms may have similar effects. The source term used by LeVeque & Yee [17],

$$u_t + u_x = \alpha u(1-u) (u-\frac{1}{2}) \qquad (5.8)$$

will not suffer so badly from this phenomenon since $u = 0$ and $u = 1$ are stable attractors hence to some degree balancing out this effect.

Conservation can not help us here unfortunately since integration around a control volume Ω will yield the term

$$\iint_{\Omega} b \ d\Omega \tag{5.9}$$

which in general will not be a difference, and will not be exactly calculable, and so the nice property ensured by conservation form in the homogeneous case will not follow.

Another complication which may arise due to source terms is stiffness (see e.g. [17]). This arises when the timestep required for stability of the pde solver (given by CFL condition, modified to allow for curved characteristics) is much larger than that required for accuracy of the ODE solver used for (3.3). Therefore we either take a small time-step and obtain an overall accurate solution at great CPU expense or a larger time-step to get an inaccurate, but stable solution. It is this last point which is the danger - i.e. there are no signs of instability to show us that we are getting an inaccurate solution. We must in fact refine Δt until successive refinements do not alter the solution values (to reasonable tolerance).

Finally we note that although an equation such as (5.4) have a well-posed analytic solution, it is well known [18] that numeric attempts at solutions can lead to chaotic or period doubling behaviour. This is an extra factor which must be taken into account when solving inhomogeneous conservation laws by whichever viewpoint.

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