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On Variational and Least Squares Methods with Adjustable Nodes

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Abstract

In the problem of residual minimisation when the nodes participate in the variations there are two possible viewpoints in carrying out the variations. We may either regard the object function as dependent on the space variables or alternatively regard the object function and the space variables as independent. The two approaches give similar variational equations in a particular illustrative example, but in general different results are obtained. The variational equations from the two approaches are compared and a particular practical advantage of using independent variations in a discrete norm discussed. An extension to more general discrete variational principles is given.

1 Introduction

The Ritz approach to the generation of discrete approximations is to use a variational or least squares formulation and restrict the admissible functions to a finite dimensional space. Normally the grid plays no part in the variational procedure but with recent interest in adaptive unstructured grids the question has been raised as to whether such variational techniques can lead to useful grids. Early attempts [1],[2] showed up the main difficulties to be the complexity of the resulting equations and the tangling of the grid. More recently, however, progress has been made in overcoming these problems and it has been shown that there are potentially considerable advantages in the approach.

Baines [3] and Tourigny and Baines [4] showed that optimal grids could be determined in this way for the problem of finding best L_2 fits to continuous functions with variable nodes in one or two dimensions, while Tourigny and Hulsemann [5] have extended the technique to variational formulations of partial differential equations with second order derivatives. A feature of these approaches is an iterative approach to the solution of the nonlinear variational equations which an optimal property of the steady Moving Finite Element (MFE) equations [7], foreseen in [1], for self-adjoint linear partial differential equations, and generated grids with the MFE method used as a (global) iterative solver which similarly has the property of reducing the variational functional for small enough steps (in the absence of singularities). These methods use variations in which the solution depends implicitly on the space variables, implying interrelated variations in both variables. A unified description which emphasises this point may be found in [8].

By contrast Roe [9], seeking the approximate solution of a simple first order partial differential equation in two dimensions using a discrete least-squares norm, has carried out variations in which the coefficients of the solution and space variables are regarded as independent variables. As in the work referred to above, the nonlinear variational equations are solved (or the least squares norm minimised) using a local iterative approach which reduces the least-squares norm. It might be expected that this approach would be a special case of the general theory in [8], but this turns out not to be the case. In the first part of this report this point is discussed and reasons given for the discrepancy.

In the second part of the report the special problem treated in [9] is generalised via the introduction of a specific discrete norm. This is followed by a description of the steepest descent iterative procedure for reducing the norm in the various cases and its effect on the crosswind diffusion.

We shall begin with an analysis of the specific problem discussed in [9] which is to find the best approximate solution and grid for the steady advection equation

$$\mathbf{a}.\nabla u = 0 \tag{1}$$

in two dimensions, with constant $\mathbf{a}$ and given inflow, using the method of least squares with piecewise linear approximation on triangles. The next two sections contrast the analysis in [8] and in [9] and the subsequent section effects a comparison.

2 First Approach

Define the functional

$$\mathcal{I} = \int_\Omega F(\mathbf{x}, u, \nabla u) d\Omega \tag{2}$$

where $u, \mathbf{x}$ are approximated by piecewise linear functions $U, \mathbf{X}$. Then, substituting $U = \sum_i U_i \psi_i$ for u and $\mathbf{X} = \sum_i \mathbf{X}_i \psi_i$ for $\mathbf{x}$ in (2), where ψ_i is the piecewise linear basis function at node i on a triangulation of the region Ω it is shown in [4],[8] that the first variation

$$\delta\mathcal{I} = \int_\Omega \sum_i \left\{ \frac{\partial F}{\partial U}\psi_i + \frac{\partial F}{\partial \nabla U}.\nabla\psi_i \right\} \delta U_i d\Omega$$

$$+ \int_\Omega \sum_i \left\{ F\nabla\psi_i + \frac{\partial F}{\partial \mathbf{X}}\psi_i - \left(\frac{\partial F}{\partial \nabla U}.\nabla\psi_i \right) \nabla U \right\} \delta\mathbf{X}_i d\Omega. \tag{3}$$

$$+\int_T \sum_i \left\{ F\nabla\psi_i + \frac{\partial F}{\partial \mathbf{X}}\psi_i - \left(\frac{\partial F}{\partial \nabla U}.\nabla\psi_i\right)\nabla U \right\} \delta\mathbf{X}_i d\Omega. \qquad (3)$$

Taking F to be the specific form

$$F(\mathbf{x}, u, \nabla u) = \frac{1}{2}(\mathbf{a}.\nabla u)^2$$

in which $\mathbf{a}$ is constant, (3) becomes

$$\delta\mathcal{I} = \sum_i \int_{T_i} (\mathbf{a}.\nabla U).(\mathbf{a}\nabla\psi_i)\delta U_i d\Omega$$

$$+\sum_i \int_{T_i} \left\{ \frac{1}{2}(\mathbf{a}.\nabla U)^2 \nabla\psi_i - (\mathbf{a}.\nabla U)(\mathbf{a}.\nabla\psi_i)\nabla U \right\} \delta\mathbf{X}_i d\Omega \qquad (4)$$

where T_i is the union of triangles abutting node i.

Since $\mathbf{a}.\nabla U$ is constant in each triangle we may write (3) as

$$\delta\mathcal{I} = \sum_i \sum_{T_i} S_T(\mathbf{a}.\nabla U)(\mathbf{a}.\nabla\psi_i)\delta U_i$$

$$+\sum_i \sum_{T_i} \left\{ \frac{1}{2} S_T(\mathbf{a}.\nabla U)^2 \nabla\psi_i - S_T(\mathbf{a}.\nabla U)(\mathbf{a}.\nabla\psi_i)\nabla U \right\} \delta\mathbf{X}_i \qquad (5)$$

where S_T is the area of triangle T_i.

Now in each such triangle

$$|\nabla\psi_i| = \frac{1}{height_i} = \frac{(opposite\ side\ length)_i}{2S_T}$$

so that

$$2S_T\nabla\psi_i = \mathbf{n}_i, \qquad 2S_T\,(\mathbf{a}.\nabla\psi_i) = \mathbf{a}.\mathbf{n}_i \qquad (6)$$

where $\mathbf{n}_i$ is the inward normal to the side of the triangle T_i opposite node i scaled so that its magnitude is the length of that side (see fig.1). Hence (5) becomes

$$\delta\mathcal{I} = \sum_i \sum_{T_i} \left(\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{a}.\mathbf{n}_i)(\delta U_i - \nabla U.\delta\mathbf{X}_i) + \frac{1}{4}(\mathbf{a}.\nabla U)^2 \mathbf{n}_i.\delta\mathbf{X}_i \right) \qquad (7)$$

and the variational equations are $\forall i$

$$\sum_{T_i} \frac{1}{2}(\mathbf{a}.\nabla U).(\mathbf{a}.\mathbf{n}_i) = 0 \qquad (8)$$

and

$$\sum_{T_i} \left\{ \frac{1}{4}(\mathbf{a}.\nabla U)^2 \mathbf{n}_i - \frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{a}.\mathbf{n}_i)\nabla U \right\} = 0. \qquad (9)$$

3 Second Approach

Alternatively, consider $\mathcal{I}$ in the discrete form

$$\mathcal{I} = \int \frac{1}{2}(\mathbf{a}.\nabla U)^2 d\Omega = \frac{1}{2}\sum_T S_T(\mathbf{a}.\nabla U)^2 \tag{10}$$

which follows since $\mathbf{a}.\nabla U$ is constant in each triangle.

Let the corners of each triangle be labelled 1,2,3 anticlockwise (see fig.1) and denote by $\sum'$ the sums over these corners. Then in each triangle

$$\begin{aligned}\nabla U &= \left(\frac{-\sum' U_1(Y_2-Y_3)}{\sum' X_1(Y_2-Y_3)}, \frac{\sum' U_1(X_2-X_3)}{-\sum' Y_1(X_2-X_3)}\right)\\ &= \left(\frac{\sum' Y_1(U_2-U_3)}{2S_T}, \frac{-\sum' X_1(U_2-U_3)}{2S_T}\right)\end{aligned} \tag{11}$$

where

$$S_T = \frac{1}{2}\sum{}' X_1(Y_2-Y_3) = -\frac{1}{2}\sum{}' Y_1(X_2-X_3) \tag{12}$$

from which it follows that

$$\mathbf{a}.\nabla U = \frac{1}{2S_T}\sum{}'(U_2-U_3)\,(aY_1-bX_1) \tag{13}$$

where $\mathbf{a} = (a,b)^T$.

Define now the so-called 'fluctuation'

$$\phi_T = \frac{1}{2}\sum{}'(U_2-U_3)\,(aY_1-bX_1) = -\frac{a}{2}\sum{}' U_1(Y_2-Y_3) - \frac{b}{2}\sum{}' U_1(X_2-X_3) \tag{14}$$

so that

$$\mathbf{a}.\nabla U = \frac{\phi_T}{S_T}. \tag{15}$$

Since $\mathbf{a}.\nabla U$ is constant in each triangle we then have from (10)

$$\mathcal{I} = \frac{1}{2}\sum_T S_T(\mathbf{a}.\nabla U)^2 = \frac{1}{2}\sum_T \frac{\phi_T^2}{S_T} \tag{16}$$

and the first variation of (16) is

$$\begin{aligned}\delta\mathcal{I} &= \frac{1}{2}\sum_i\sum_{T_i}\left\{\frac{2\phi_T}{S_T}\frac{d\phi_T}{dU_i}\delta U_i + \left(\frac{2\phi_T}{S_T}\frac{d\phi_T}{d\mathbf{X}_i} - \frac{\phi_T^2}{S_T^2}\frac{dS_T}{d\mathbf{X}_i}\right).\delta\mathbf{X}_i\right\}\\ &= \frac{1}{2}\sum_i\sum_{T_i}\frac{2\phi_T}{S_T}\left(-\frac{1}{2}a(Y_2-Y_3) - \frac{1}{2}b(X_2-X_3)\right)\delta U_i\end{aligned}$$

$$+\frac{1}{2}\sum_i\sum_{T_i}\left\{\frac{2\phi_T}{S_T}(U_2-U_3)\begin{pmatrix}-b\\a\end{pmatrix}-\frac{\phi_T^2}{S_T^2}\begin{pmatrix}Y_2-Y_3\\-(X_2-X_3)\end{pmatrix}\right\}.\delta\mathbf{X}_i \tag{17}$$

using (11) and (12).

Thus the variational equations are

$$\sum_{T_i}\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{a}.\mathbf{n}_i)=0 \tag{18}$$

as in (8), and

$$\frac{1}{2}\sum_{T_i}\left(-\frac{\phi_T}{S_T}(U_3-U_2)\begin{pmatrix}-b\\a\end{pmatrix}+\frac{1}{2}\frac{\phi_T^2}{S_T^2}\mathbf{n}_i\right)=0$$

or

$$\sum_{T_i}\left\{\frac{1}{4}(\mathbf{a}.\nabla U)^2\mathbf{n}_i-\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{t}_i.\nabla U)\begin{pmatrix}-b\\a\end{pmatrix}\right\}=0 \tag{19}$$

where, in each triangle T_i the convention is that node i counts as node 1 and $\mathbf{t}_i$ is the vector from node 2 to node 3, corresponding to the side of the triangle opposite node 1 (see fig.1 and [8]).

4 Comparison of the two Approaches

The variational equations (8) and (18) are the same in both approaches. On the other hand, comparing (9) and (19) we see that they differ in the second terms, which are in the (distinct) directions of ∇U and $(-b,a)^T$ respectively. These directions coincide only when $\mathbf{a}.\nabla U=0$ (which is satisfied by the exact solution). Moreover, the ratio of the magnitudes of the second terms in (9) and (19) is

$$\left|\frac{(\mathbf{a}.\mathbf{n}_i)\nabla U}{(\mathbf{t}_i.\nabla U)(-b,a)}\right|$$

$$=\frac{|\mathbf{a}|\,|\mathbf{n}_i|\cos\theta\,|\nabla U|}{|\mathbf{t}_i|\,|\nabla U|\cos\phi\,|(-b,a)|} \tag{20}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{n}_i$ and ϕ is the angle between $\mathbf{t}_i$ and ∇U. Since $|\mathbf{a}|=|(-b,a)|$ and $|\mathbf{n}_i|=|\mathbf{t}_i|$ this ratio reduces to

$$\frac{\cos\theta}{\cos\phi}. \tag{21}$$

Since $\mathbf{n}_i$ is perpendicular to $\mathbf{t}_i$, if also $\mathbf{a}.\nabla U=0$ so that ∇U is perpendicular to $\mathbf{a}$, this ratio is unity. We see therefore that the *coefficients* of $(\mathbf{a}.\nabla U)$ in the second terms in (9) and (19) coincide when $\mathbf{a}.\nabla U=0$ (i.e. when U satisfies the exact eqaution).

The difference between the variational equations (9) and (19) originates in the two contrasting approaches. In the first more general approach U is dependent on $\mathbf{X}$ when carrying out the variations, but in the second approach the U_i and $\mathbf{X}_i$ are treated as independent variables. In the former case the ∇U term in (9) arises naturally from Lagrangian variations of the form

$$Du = \partial u + \nabla u . D\mathbf{x}, \tag{22}$$

while in the latter case the $(-b, a)^T$ term in (19) arises from differentiating the fluctuation ϕ_T in the discrete norm (16) with respect to the (independent) variables $\mathbf{X}_i$ $(\forall i)$.

It turns out that there is some practical advantage in taking the second approach. We therefore develop an extension of (10) to more general variational problems.

5 A General Discrete Variational Principle

Consider again the variational problem

$$\min_{u,\mathbf{x}} \int_\Omega F(\mathbf{x}, u, \nabla u) d\Omega \tag{23}$$

where $u \sim U$, and $\mathbf{x} \sim \mathbf{X}$ are piecewise linear in each triangle. If U is made to depend on $\mathbf{X}$ in carrying out the minimisation the first variation is as in (3). However, by replacing (23) with a discrete norm we may treat the coefficients of U and $\mathbf{X}$ independently, as in section 3, and obtain a generalisation of the second approach above.

We therefore introduce the discrete variational problem

$$\min_{U,\mathbf{X}} \sum_T \frac{1}{3} S_T \sum_{j=1}^{3} F(\mathbf{X}_j, U_j, \nabla U) d\Omega, \tag{24}$$

where j runs over the corners of the triangle T, *as if* F were projected into the space of piecewise linear functions in each triangle and exactly integrated in (23). (Note that a similar discrete form was originally used by Euler in his approach to the continuous case in one dimension [10].)

The discrete functional is

$$I = \sum_T \frac{1}{3} S_T \sum_{j=1}^{3} F(\mathbf{X}_j, U_j, \nabla U) \tag{25}$$

(cf. (10)). Using the same notation as before, the first variation of I (in which U_i and $\mathbf{X}_i = (X_i, Y_i)$ are varied independently) is

$$\delta I = \sum_i \sum_{T_i} \frac{1}{3} S_T \left\{ \frac{\partial F}{\partial U_i} + \sum_{j=1}^{3} \left(\frac{\partial F}{\partial \nabla U} \right)_j . \frac{\partial \nabla U}{\partial U_i} \right\} \delta U_i$$

$$+\sum_i\sum_{T_i}\left\{\frac{1}{6}(Y_2-Y_3)\sum_{j=1}^{3}(F)_j+\frac{1}{3}S_T\frac{\partial F}{\partial X_i}+\frac{1}{3}\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\cdot\begin{pmatrix}0\\ -\frac{1}{2}(U_2-U_3)\end{pmatrix}\right.$$

$$\left.-\frac{1}{6}(Y_2-Y_3)\left(\nabla U.\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\right)\right\}\delta X_i$$

$$+\sum_i\sum_{T_i}\left\{-\frac{1}{6}(X_2-X_3)\sum_{j=1}^{3}(F)_j+\frac{1}{3}S_T\frac{\partial F}{\partial Y_i}+\frac{1}{3}\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\cdot\begin{pmatrix}\frac{1}{2}(U_2-U_3)\\ 0\end{pmatrix}\right.$$

$$\left.+\frac{1}{6}(X_2-X_3)\left(\nabla U.\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\right)\right\}\delta Y_i \tag{26}$$

using both (11) and (12), where again in each triangle node i counts as node 1 in the numbering convention used. Hence the variational equations at each node i are

$$\sum_{T_i}\frac{1}{3}S_T\left\{\frac{\partial F}{\partial U_i}+\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\cdot\frac{\partial \nabla U}{\partial U_i}\right\}=0 \tag{27}$$

together with the vector equation

$$\sum_{T_i}\left\{-\frac{1}{6}\sum_{j=1}^{3}(F)_j\mathbf{n}_i+\frac{1}{3}S_T\frac{\partial F}{\partial \mathbf{X}_i}-\frac{1}{6}(U_3-U_2)\sum_{i=1}^{3}\left(-\frac{\partial F}{\partial U_y},\frac{\partial F}{\partial U_x}\right)_j^T\right.$$

$$\left.+\frac{1}{6}\left(\nabla U.\sum_{i=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\right)\mathbf{n}_i\right\}=0. \tag{28}$$

From the variations in (3) we should compare equation (27) with

$$\int_\Omega\left\{\frac{\partial F}{\partial U}\psi_i+\frac{\partial F}{\partial \nabla U}.\nabla\psi_i\right\}d\Omega=0 \tag{29}$$

(of which it is an obvious discretisation), and (28) with

$$\int_\Omega\left\{F\nabla\psi_i+\frac{\partial F}{\partial \mathbf{X}}\psi_i-\left(\frac{\partial F}{\partial \nabla U}.\nabla\psi_i\right)\nabla U\right\}d\Omega=0 \tag{30}$$

where the connection is far less straightforward, the discrepancy arising from the approach as well as the use of the discrete variational principle.

In the particular case when $F=\frac{1}{2}(\mathbf{a}.\nabla u)^2$ equation (27) reduces to

$$\sum_{T_i}\frac{1}{2}\left(\mathbf{a}.\mathbf{n}_i\right)\left(\mathbf{a}.\nabla U\right)=0 \tag{31}$$

as in (8),(18) while equation (28) becomes

$$\sum_{T_i}\left\{-\frac{1}{4}(\mathbf{a}.\nabla U)^2\mathbf{n}_i-\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{t}_i.\nabla U)\begin{pmatrix}-b\\ a\end{pmatrix}+\frac{1}{2}(\mathbf{a}.\nabla U)^2\mathbf{n}_i\right\}$$

$$=\sum_{T_i}\left\{\frac{1}{4}(\mathbf{a}.\nabla U)^2\mathbf{n}_i-\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{t}_i.\nabla U)\begin{pmatrix}-b\\ a\end{pmatrix}\right\}=0, \qquad (32)$$

as in (19).

In the important special case $F=\frac{1}{2}|\nabla u|^2+f(\mathbf{x})u$, which corresponds to u satisfying Poisson's equation $-\nabla^2 u = f(\mathbf{x})$ the variational equations at node i are

$$\sum_{T_i}\left\{\frac{1}{3}S_T f(\mathbf{X})_i+\frac{1}{2}(\nabla U)_i.\mathbf{n}_i\right\}=0 \qquad (33)$$

plus

$$\sum_{T_i}\left\{\frac{1}{4}|\nabla U|^2\mathbf{n}_i+\frac{1}{3}S_T U_i\frac{df}{d\mathbf{X}_i}-\frac{1}{2}(\mathbf{t}_i.\nabla U)\begin{pmatrix}-U_y\\ U_x\end{pmatrix}\right\}=0. \qquad (34)$$

6 Steepest Descent Algorithms

A feature of the analysis in [9] is the solution of the variational equations (18) and (19) by a method of steepest descent of the form

$$\delta U_i=-\sum_{T_i}\frac{1}{2}(\mathbf{a}.\mathbf{n}_i)(\mathbf{a}.\nabla U)\delta\tau \qquad (35)$$

$$\delta\mathbf{X}_i=-\sum_{T_i}\left\{\frac{1}{4}(\mathbf{a}.\nabla U)^2\mathbf{n}_i-\frac{1}{2}(\mathbf{a}.\nabla U)(\mathbf{t}_i.\nabla U)\begin{pmatrix}-b\\ a\end{pmatrix}\right\}\delta\sigma \qquad (36)$$

where $\delta\tau$ and $\delta\sigma$ are (sufficiently small) relaxation factors.

An upwind version of (35) may also be devised in which the term $\mathbf{a}.\mathbf{n}_i$ is replaced by $max\,(0,\mathbf{a}.\mathbf{n}_i)$ but at the expense of losing the monotonicity principle. The latter method is very close to the LDA multidimensional upwind method.

A significant aspect of (36) is that $\delta\mathbf{X}_i$ is made up of two contributions, one a vector in the direction $(-b,a)^T$ and the other a weighted average of normals $\mathbf{n}_i$ (the former at right angles to the characteristic direction).

An important relevant property of piecewise linear approximation on triangles is as follows. When a side of a triangle aligns with the characteristic direction, with the end nodes of the side taking the same U value, then because the gradient of U is perpendicular to that side the residual $\mathbf{a}.\nabla U$ in the triangle vanishes so that the contribution to the norm from the triangle is automatically minimised.

The order of the operations in the steepest descent method is not prescribed nor is the size of the relaxation factors. In one particular scenario, however, an apparently optimal choice may be devised. Suppose that in the initial distribution the value of U is zero everywhere except at inflow points so that the residual is also zero everywhere except on elements adjacent to the inflow. Then one step of (35) will spread the inflow values into the interior, favouring the characteristic direction but including some crosswind diffusion, resulting in non-zero residuals in all cells adjacent to the inflow. Suppose also that this is followed by a step of (36) in which the movement of the nodes has a component perpendicular to the characteristic direction which reduces the residual. The effect is to drive the smaller residuals which border the main flow arising from the crosswind diffusion towards zero, effectively annihilating such diffusion. Repeating these two steps will eventually drive the inflow disturbance in the characteristic direction, at each stage minimising the crosswind diffusion. In this way the characteristicdirectional properties are respected, which makes the procedure particularly suitable for hyperbolic problems. The discussion is borne out by the demonstration in fig.2 in which the characteristics are circles and there is just one non-zero inflow point in the middle of the left half of the base. Note the 'buffer zone' (two circular strips) between the non-zero characteristic and its neighbours, outside which there is effectively no diffusion.

A comparable steepest descent method for the first approach would be (35) followed by

$$\delta \mathbf{X}_i = -\sum_{T_i} \left\{ \frac{1}{4} (\mathbf{a}.\nabla U)^2 \mathbf{n}_i - \frac{1}{2} (\mathbf{a}.\nabla U)(\mathbf{a}.\mathbf{n}_i) \nabla U \right\} \delta\sigma. \tag{37}$$

The point to notice here is that in (37) the second term is not in the direction $(-b, a)^T$ but is a weighted average of gradients ∇U. This makes it less suitable as a streamwise method, at least in the mode described in the previous paragraph, since initially the gradients are unlikely to have the coherence possessed by $(-b, a)^T$ in the previous approach.

In the general case (2) the steepest descent steps are given by

$$\delta U_i = -\int_\Omega \left\{ \frac{\partial F}{\partial U} \psi_i + \frac{\partial F}{\partial \nabla U} . \nabla \psi_i \right\} d\Omega \delta\tau \tag{38}$$

and

$$\delta \mathbf{X}_i = -\int_\Omega \left\{ F \nabla \psi_i + \frac{\partial F}{\partial \mathbf{X}} \psi_i - \left(\frac{\partial F}{\partial \nabla U} . \nabla \psi_i \right) \nabla U \right\} d\Omega \delta\sigma \tag{39}$$

while in the discrete general case (24) they are

$$\delta U_i = -\sum_{T_i} \frac{1}{3} S_T \left\{ \frac{\partial F}{\partial U_i} + \sum_{j=1}^{3} \left(\frac{\partial F}{\partial \nabla U} \right)_j . \frac{\partial \nabla U}{\partial U_i} \right\} \delta\tau \tag{40}$$

and

$$\delta \mathbf{X}_i = -\sum_{T_i} \left\{ -\frac{1}{6} \sum_{j=1}^{3} (F)_j \mathbf{n}_i + \frac{1}{3} S_T \frac{\partial F}{\partial \mathbf{X}_i} - \frac{1}{6} (U_3 - U_2) \sum_{j=1}^{3} \left(-\frac{\partial F}{\partial U_y}, \frac{\partial F}{\partial U_x} \right)_j^T \right.$$

$$+\frac{1}{6}\left(\nabla U.\sum_{j=1}^{3}\left(\frac{\partial F}{\partial \nabla U}\right)_j\right)\mathbf{n}_i\Bigg\}\,\delta\sigma. \qquad (41)$$

Once again there is a difference between the grid update steps in the two approaches, most significantly in the terms proportional to ∇U in (39) and those proportional to $\left(-\frac{\partial F}{\partial U_y}, \frac{\partial F}{\partial U_x}\right)$ in (41).

7 Conclusions

We have seen that in carrying out residual minimisation with variable nodes the general theory in [8] does not reduce to the special case studied in [9]. The principal reason is that in the general theory the solution U is dependent on the grid description $\mathbf{X}$ and varies as the grid adjusts in a Lagrangian manner, while in the discrete approach of [9] the nodal coordinates U_i and $\mathbf{X}_i$ are varied independently. The distinction between the two approaches is complicated by the fact that in the special problem considered in [9] the continuous and discrete variational principles are identical.

The discrete norm has been extended to general variational principles and the corresponding variational equations obtained. A steepest descent strategy has been described either as an optimisation procedure for the variational principle or as an iterative procedure for solving the variational equations. A geometrical interpretation is given which throws light on the optimal ordering in the steepest descent method to give a solution with minimal crosswind diffusion.

8 Acknowledgement

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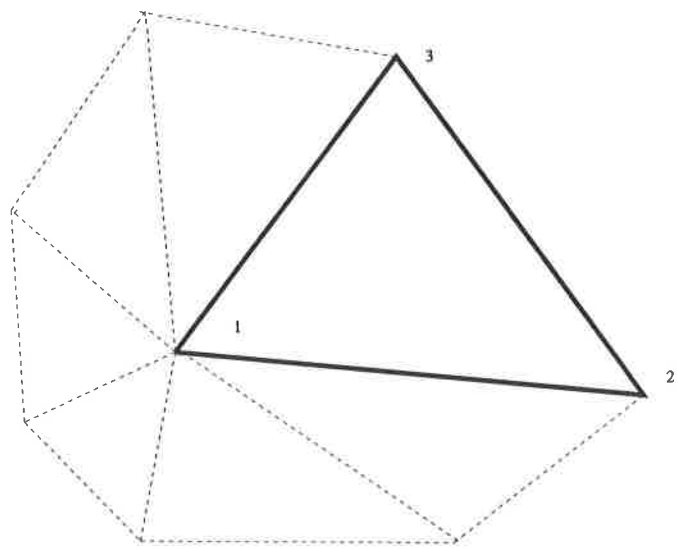
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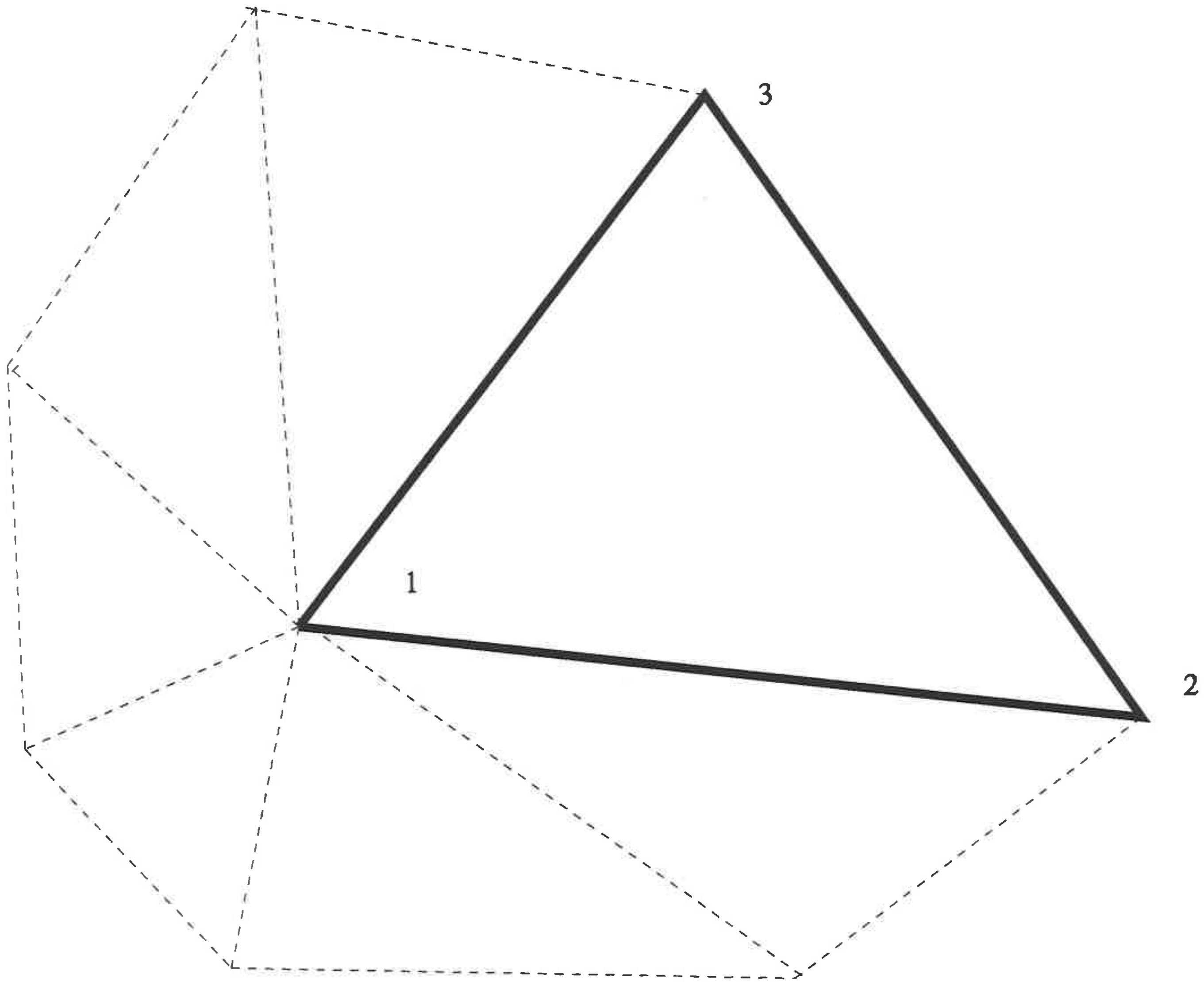
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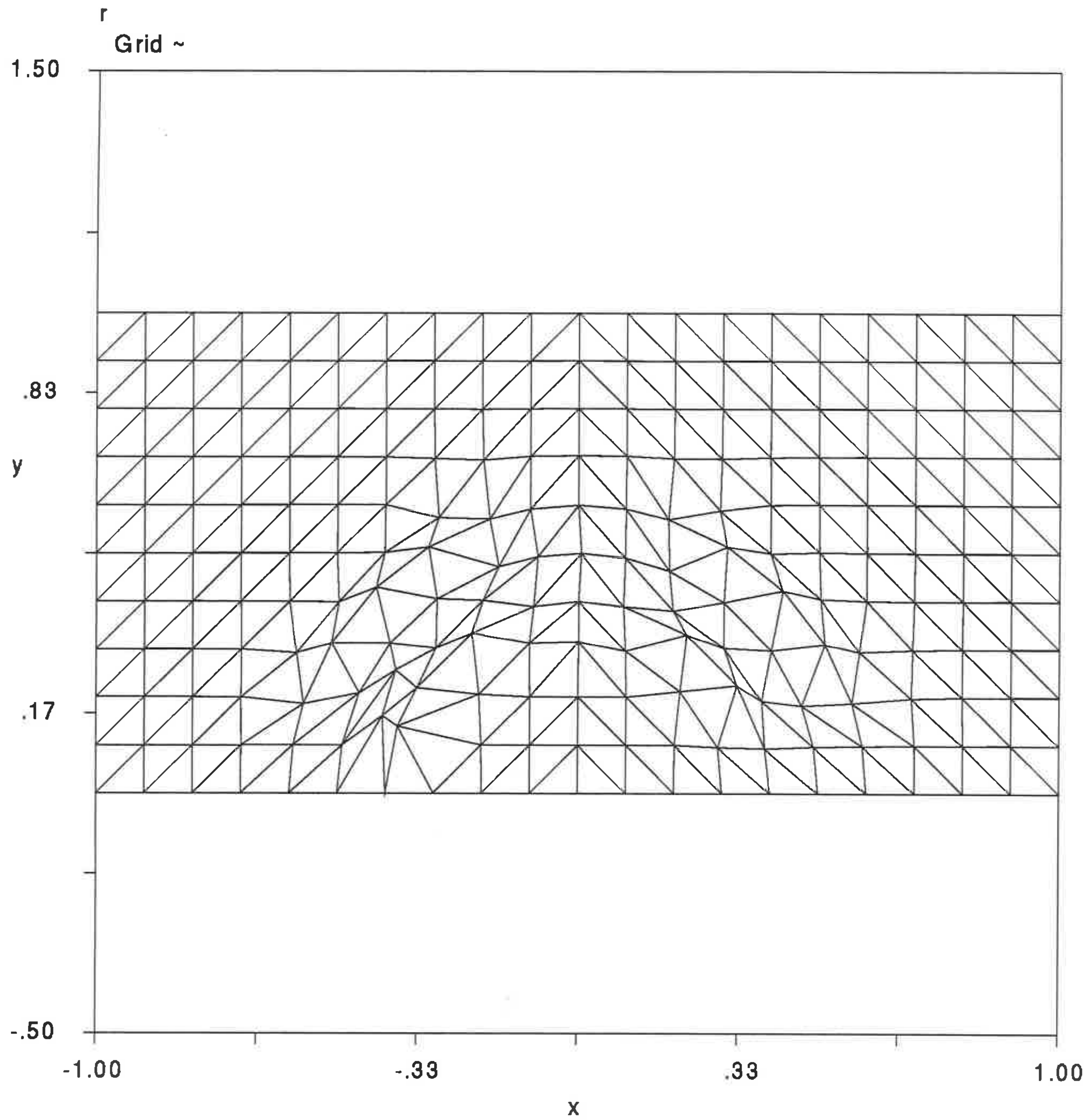

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