SINGULAR VALUE AND GENERALISED SINGULAR VALUE DECOMPOSITIONS AND THE SOLUTION OF LINEAR MATRIX EQUATIONS

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by

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Abbreviated Title:

SVD, GSVD and Linear Matrix Equations.

Abstract:

The solution of the linear matrix equations (i) AXB + CYD = E and (ii) (AXB,FXG) = (E,H) are considered. New necessary and sufficient conditions for the consistency of the equations are derived, some using the generalized singular value decomposition. Special cases (iii) AX + YD = E and (iv) AXB = E are treated using the singular value decomposition. Numerical algorithms for the solutions are also suggested.

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1. Introduction.

Let $\mathbb{R}^{m\times n}$ denote the space of real $m \times n$ matrices.

We consider the solution of the linear matrix equations

$$AXB + CYD = E \tag{1}$$

with $A \in \mathbb{R}^{m\times p}$, $B \in \mathbb{R}^{q\times n}$, $C \in \mathbb{R}^{m\times r}$ and $D \in \mathbb{R}^{s\times n}$;

and

$$\begin{cases} AXB = E , & (2a) \\ FXG = H , & (2b) \end{cases}$$

with $F \in \mathbb{R}^{\tilde{m}\times p}$ and $G \in \mathbb{R}^{q\times \tilde{n}}$.

We also consider the special cases

$$AX + YD = E \tag{3}$$

and

$$AXB = E \tag{4}$$

of equations (1) and (2) respectively.

For equation (3), the necessary and sufficient conditions for its consistency have been derived in [1] [9] (and some references therein). For inconsistent equations, ℓ_p and Chebyshev solutions can be considered [11] [12]. For equation (4), the consistency conditions were given by Penrose (see [7]). If equation (3) or (4) is consistent, a solution can be obtained using generalized inverses (GI) [1] [7]. In sections 4 and 5, the singular value decomposition (SVD) [4] will be used to investigate equations (3) and (4), and a simple and clear exposition, in terms of consistency conditions, analytic and numerical solutions, and ℓ_2-solutions, will be given.

For the general cases in (1) and (2), consistency conditions were given in [2] and [6] respectively, and solutions were again given in terms of GI . Simpler conditions for consistency of equations (1) and (2) can be obtained, through the use of the generalized singular value decomposition (GSVD) ([4] [8] [10] and references therein), and stable numerical algorithms for their solutions, arise naturally; the results will be contained in sections 3 and 6 respectively.

Many authors were not aware of the fact that equations (1) and (2) are dual in some sense. The duality is discussed in section 7.

The paper is completed with a brief introduction to SVD and GSVD in section 2, and a conclusion in section 8.

Note that an excellent thesis on the general equation

$$\sum_{j=1}^{N} A_{ij}X_jB_{ij} = C_i , \quad i=1,...,M ;$$

can be found in [5]. Questions of consistency, near-consistency (an equation was defined to be ε-consistent iff the residual is less than ε), and applications to output feedback pole assignment problems in control theory, were considered. Some results in sections 3 and 6 have been obtained (in less elegant forms) in [5], as GSVD have been used unknowingly.

Other linear matrix equations have been considered by the author in [3] and some more results will appear in future papers.

2. SVD and GSVD. [4] [8] [10]

Given a matrix $A \in \mathbb{R}^{m\times n}$ of rank k, one has

$$A = UDV^T \tag{5}$$

where $U = (U_1, U_2)$ and $V = (V_1, V_2)$ are orthogonal matrices, with

$$D = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$, $\sigma_i > 0$.

It is easy to see that the matrices U_1, V_1, V_2 and U_2 span the range and null spaces of A and A^T respectively. Other properties of SVD can be found in standard texts such as [4], and references therein.

The GSVD, a generalization of SVD, can be described as follows:- [8]

Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times n}$ with equal number of columns, there exists orthogonal matrices U and V, and non-singular X so that

$$A = U\Sigma_A X, \quad B = V\Sigma_B X; \tag{7}$$

where $\Sigma_A \in \mathbb{R}^{m\times n}$, $\Sigma_B \in \mathbb{R}^{p\times n}$, and $k = \mathrm{rank}\ (C) = \mathrm{rank} \begin{pmatrix} A \\ B \end{pmatrix}$, with

$$\Sigma_A = \left(\begin{array}{ccc:c} I_A & & & \\ & S_A & & 0 \\ & & O_A & \end{array}\right), \tag{8a}$$

$$\begin{array}{cccc} r & s & k-r-s & n-k \end{array}$$

$$\Sigma_B = \left(\begin{array}{ccc|c} 0_B & & & \\ & S_B & & 0 \\ & & I_B & \end{array} \right) ; \tag{8b}$$

$$\begin{array}{cccc} r & s & k-r-s & n-k \end{array}$$

I_A and I_B are identity matrices, 0_A and 0_B zero matrices, and

$$S_A = \mathrm{diag}(\alpha_1,\ldots,\alpha_s) , \tag{9a}$$

$$S_B = \mathrm{diag}(\beta_1,\ldots,\beta_s) , \tag{9b}$$

with $1 > \alpha_1 \geqq \ldots \geqq \alpha_s > 0$, $0 < \beta_1 \leqq \ldots \leqq \beta_s < 1$,

and $\alpha_i^2 + \beta_i^2 = 1, \quad i=1,\ldots,s$.

Some submatrices in equation (8) can vanish, depending on the structures of matrices A and B .

Proofs and properties concerning the GSVD can be found in [4] [8] [10] and references therein.

In situations where the matrix X in equation (7) has to be inverted, (e.g. in sections 3 and 6), ill-conditioning may occur, as the matrix X is not orthogonal. In [8], the matrix can be expressed as

$$X = Q \begin{pmatrix} R^{-1}W & 0 \\ 0 & I \end{pmatrix} \tag{10}$$

where the matrices Q and W are orthogonal, and

$$C = \begin{pmatrix} A \\ B \end{pmatrix} = P \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} Q^T , \tag{11}$$

with the matrix P being orthogonal. (It can be the SVD in equation (11).)

Thus, the matrix X will be ill-conditioned if the smallest non-zero singular value of C is small, i.e. when the (numerical) rank determination of the matrix C is not straightforward.

A stable numerical algorithm for the computation of the GSVD by Stewart can be found in [10].

3. $AXB + CYD = E$.

Decomposing the matrix-pairs (A^T, C^T) and (B,D) using GSVD, equation (1) is equivalent to

$$X_1^T \Sigma_A^T U_1^T \cdot X \cdot U_2 \Sigma_B X_2 + X_1^T \Sigma_C^T V_1^T \cdot Y \cdot V_2 \Sigma_D X_2 = E \ , \tag{12}$$

where the matrices U_i and V_i are orthogonal, and X_i are non-singular, as in equations (7) to (9).

Define $\tilde{X} = U_1^T X U_2$, $\tilde{Y} = V_1^T Y V_2$ and $\tilde{E} = X_1^{-T} E X_2^{-1}$, equation (12) now reads

$$\Sigma_A \tilde{X} \Sigma_B + \Sigma_C \tilde{Y} \Sigma_D = \tilde{E} \ . \tag{13}$$

Note that transforming equation (12) to (13) does not change the equation's consistency.

Partitioning matrices $\tilde{X}, \tilde{Y}$ and $\tilde{E}$ according to that of the Σ's , equation (13) is equivalent to

$$\begin{pmatrix} I_A & & \\ & S_A & \\ & & 0_A^T \\ \hdashline & 0 & \end{pmatrix} \tilde{X} \begin{pmatrix} I_B & & & \vdots & \\ & S_B & & \vdots & 0 \\ & & 0_B & \vdots & \end{pmatrix}$$

$$+ \begin{pmatrix} 0_C^T & & \\ & S_C & \\ & & I_C \\ \hdashline & 0 & \end{pmatrix} \tilde{Y} \begin{pmatrix} 0_D & & & \vdots & \\ & S_D & & \vdots & 0 \\ & & I_D & \vdots & \end{pmatrix} = \tilde{E}$$

(c.f. equation (8))

$$\Longleftrightarrow \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12}S_B & 0 & 0 \\ S_A\tilde{X}_{21} & S_A\tilde{X}_{22}S_B+S_C\tilde{Y}_{22}S_D & S_C\tilde{Y}_{23} & 0 \\ 0 & \tilde{Y}_{32}S_D & \tilde{Y}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{E}$$

$$= \left(\begin{array}{ccc:c} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & \\ \tilde{E}_{21} & \tilde{E}_{22} & \tilde{E}_{23} & \tilde{E}_{\bullet 4} \\ \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} & \\ \hdashline & \tilde{E}_{4\bullet} & & \tilde{E}_{44} \end{array}\right) . \qquad (14)$$

The following theorem is a consequence of equation (14):

Theorem 1. Equation (1) is consistent if and only if the following submatrices of $\tilde{E}$ vanish:-

$\tilde{E}_{13}$, $\tilde{E}_{31}$, $\tilde{E}_{4\bullet}$, $\tilde{E}_{\bullet 4}$, $\tilde{E}_{44}$.

For consistent equations, the submatrices

$\tilde{X}_{13}$, $\tilde{X}_{23}$, $\tilde{X}_{33}$, $\tilde{X}_{32}$, $\tilde{X}_{31}$; $\tilde{Y}_{13}$, $\tilde{Y}_{12}$, $\tilde{Y}_{11}$, $\tilde{Y}_{21}$, $\tilde{Y}_{31}$;

of $\tilde{X}$ and $\tilde{Y}$ respectively are arbitrary, and additional degrees of freedom can be found in

$\tilde{X}_{22} = (\tilde{x}_{ij})$ and $\tilde{Y}_{22} = (\tilde{y}_{ij})$.

Elementwise, one has

$$\begin{pmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{pmatrix} = M^{+}_{ij}\tilde{e}_{ij} + (I-M^{+}_{ij}M_{ij})Z_{ij} \qquad (15a)$$

where $S_A = \mathrm{diag}(\alpha_i)$, $S_B = \mathrm{diag}(\beta_i)$, $S_C = \mathrm{diag}(\gamma_i)$,

$S_D = \mathrm{diag}(\delta_i)$, and $\tilde{E} = (\tilde{e}_{ij})$;

$M_{ij} = (\alpha_i\beta_j \ , \ \gamma_i\delta_j)$, $(\cdot)^{+}$ denoting the (1,2,3,4)- or Penrose - GI [7], and the vectors Z_{ij} are arbitrary.

Or consider $\tilde{Y}_{22}$ to be arbitrary, $\tilde{X}_{22}$ must then be chosen to be

$$\tilde{X}_{22} = S_A^{-1}(\tilde{E}_{22}-S_C\tilde{Y}_{22}S_D)S_B^{-1} . \tag{15b}$$

(Or consider $\tilde{X}_{22}$ to be arbitrary and choose $\tilde{Y}_{22}$ similar to equation (15b) accordingly.)

(Proof): The consistency conditions, the arbitrariness of the submatrices and equation (15b) are trivial from equation (14).

For $\tilde{X}_{22}$ and $\tilde{Y}_{22}$, consider the (i,j) - component $\tilde{x}_{ij}$ and $\tilde{y}_{ij}$, and equation (14) implies

$$M_{ij}\begin{pmatrix}\tilde{x}_{ij}\\ \tilde{y}_{ij}\end{pmatrix} = \tilde{e}_{ij}$$

and thus equation (15a). Note that, from the definition of GSVD and equation (9), the row vectors M_{ij} are non-zero and right-invertible.

■

Using Theorem 1 and equation (15b), one characterization of the solution of a consistent equation (1) will be

$$X = U_1\begin{pmatrix}\tilde{E}_{11} & \tilde{E}_{12}S_B^{-1} & Z_2\\ S_A^{-1}\tilde{E}_{21} & S_A^{-1}(\tilde{E}_{22}-S_CZ_1S_D)S_B^{-1} & Z_3\\ Z_4 & Z_5 & Z_6\end{pmatrix}U_2^T ,$$

and

$$Y = V_1 \begin{bmatrix} Z_7 & Z_8 & Z_9 \\ Z_{10} & Z_1 & S_C^{-1}\tilde{E}_{23} \\ Z_{11} & \tilde{E}_{32}S_D^{-1} & \tilde{E}_{33} \end{bmatrix} V_2^T ,$$

where the matrices Z_1 to Z_{11} are arbitrary.

Equation (14) leads naturally to a numerical algorithm for the solution of a consistent equation (1). The process will then be numerically unstable (and equation (1) numerically ill-conditioned) if any of the generalized singular values (GSV) α_i , β_i , γ_i and δ_i is small, or the matrix X_1 or X_2 ill-conditioned. (c.f. discussion after equation (11).) Small GSV may have to be reset to zero, with the resulted errors treated as residuals of equation (1).

The transformations by unitary matrices of $\tilde{X}$ and $\tilde{Y}$ back to the solution X and Y will be well-conditioned [4].

By choosing various arbitrary matrices (ignoring equation (15b)) to be zero, a solution of least 2- or F-norm may be obtained, but the residual of an inconsistent equation (1) will not be of least norm, as X_1 and X_2 in equations (12) and (13) are not orthogonal.

4. $AX + YD = E$.

For the special case (3) of equation (1), it is not necessary to use GSVD for the solution. (The GSVD of (A,I) degenerates into the SVD of A.)

Decomposing the matrices A and D by SVD, equation (3) is equivalent to

$$U_A D_A V_A^T \cdot X + Y \cdot U_D D_D V_D^T = E$$

$$\Longleftrightarrow \quad D_A \tilde{X} + \tilde{Y} D_D = \tilde{E} \tag{16}$$

with $\tilde{X} = V_A^T X$, $\tilde{Y} = YU_D$ and $\tilde{E} = U_A^T E V_D$.

Note that all the transformations involved in equation (16) are orthogonal.

Partitioning equation (16), according to that of D_A and D_D, yields

$$\begin{pmatrix} \Sigma_A & 0 \\ 0 & 0 \end{pmatrix} \tilde{X} + \tilde{Y} \begin{pmatrix} \Sigma_D & 0 \\ 0 & 0 \end{pmatrix} = \tilde{E}$$

$$\Longleftrightarrow \quad \begin{pmatrix} \Sigma_A \tilde{X}_{11} + \tilde{Y}_{11} \Sigma_D & \Sigma_A \tilde{X}_{12} \\ \tilde{Y}_{21} \Sigma_D & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}. \tag{17}$$

From equation (17) one has the following theorem:

<u>Theorem 2.</u> Equation (3) is consistent if and only if

$$\tilde{E}_{22} = U_{A2}^T E V_{D2} = 0, \tag{18}$$

with $U_A = (U_{A1}, U_{A2})$ and $V_D = (V_{D1}, V_{D2})$.

For a consistent equation (3), the submatrices $\tilde{X}_{21}$, $\tilde{X}_{22}$, $\tilde{Y}_{12}$, $\tilde{Y}_{22}$ of $\tilde{X}$ and $\tilde{Y}$ are arbitrary, and additional degrees of freedom can be found in

$$\tilde{X}_{11} = (\tilde{x}_{ij}) \quad \text{and} \quad \tilde{Y}_{11} = (\tilde{y}_{ij}) .$$

Elementwise, one has

$$\begin{pmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{pmatrix} = M^{+}_{ij}\tilde{e}_{ij} + (I-M^{+}_{ij}M_{ij})Z_{ij} \tag{19a}$$

where $\Sigma_A = \mathrm{diag}(\alpha_i)$, $\Sigma_D = \mathrm{diag}(\delta_i)$, $\tilde{E} = (\tilde{e}_{ij})$,

$M_{ij} = (\alpha_i,\delta_i)$ and Z_{ij} are arbitrary.

Or consider $\tilde{Y}_{11}$ to be arbitrary and $\tilde{X}_{11}$ has to be chosen to be

$$\tilde{X}_{11} = \Sigma_A^{-1}(\tilde{E}_{11} - \tilde{Y}_{11}\Sigma_D) . \tag{19b}$$

If condition (18) is satisfied, the solution of equation (3) may be written as

$$X = V_A \begin{pmatrix} \Sigma_A^{-1}(\tilde{E}_{11}-Z_1\Sigma_D) & \Sigma_A^{-1}\tilde{E}_{12} \\ Z_2 & Z_3 \end{pmatrix}$$

$$= V_A \begin{pmatrix} \Sigma_A^{-1} \\ 0 \end{pmatrix} U_{A1}^T(E-U_{A1}Z_1\Sigma_D V_{D1}^T)V_D + V_{A2}(Z_2,Z_3) \tag{20a}$$

and

$$Y = \begin{pmatrix} Z_1 & Z_4 \\ \tilde{E}_{21}\Sigma_D^{-1} & Z_5 \end{pmatrix} U_D^T = \begin{pmatrix} Z_1 & Z_4 \\ U_{A2}^T E V_{D1}\Sigma_D^{-1} & Z_5 \end{pmatrix} U_D^T , \tag{20b}$$

where the matrices Z_1 to Z_5 are arbitrary.

Condition (18) and equation (20) are equivalent to those given in equations (2) to (4) in [1], without any explicit use of GI.

Again, equation (17) leads naturally to a numerical algorithm for the solution of equation (3) - if it is consistent. For inconsistent systems, a least square type solution can also be found from equations (17) and (19a), as all the transformations involved are orthogonal. The size of the residual will be the same as that of $\tilde{E}_{22}$.

The algorithm will be numerically unstable when any of the SV α_i and δ_i is small. Again, small SV can be reset to zero and the resulted errors treated as residuals of equation (3).

The SVD used to transform equation (3) to (17) can be replaced by the less expensive QR decompositions [4], if the rank determinations of the matrices A^T and D are straightforward, and equation (3) is then equivalent to

$$R_A^T Q_A^T X + Y Q_D R_D = E$$

$$\Longleftrightarrow \quad R_A^T \tilde{X} + \tilde{Y} R_D = E , \tag{21}$$

where $\tilde{X} = Q_A^T X$, $\tilde{Y} = Y Q_D$, with the matrices Q_A and Q_D being orthogonal, and R_A and R_D being upper triangular or trapezoidal.

An equivalent theory of consistency can be derived from equation (21), instead of (17). A numerical algorithm for the solution of equation (3) can then be derived, by considering the individual component of (21) in a row-wise or column-wise fashion.

Note also that special cases of equation (1), e.g.

$$AXB + Y = E , \tag{22}$$

can be treated using SVD in a similar fashion as in this section.

5. $AXB = E$.

It is obvious that consistency for equations (2a) and (2b), (both in the form of equation (4)) is necessary for the consistency of equation (2).

To study the consistency of equation (4), we decompose the matrices A and B by SVD:

$$U_A D_A V_A^T \cdot X \cdot U_B D_B V_B^T = E$$

$$\Longleftrightarrow \begin{pmatrix} \Sigma_A & 0 \\ 0 & 0 \end{pmatrix} \tilde{X} \begin{pmatrix} \Sigma_B & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix} = \tilde{E} , \qquad (23)$$

where $\tilde{X} = V_A^T X U_B$ and $\tilde{E} = U_A^T E V_B$.

Equation (23) gives rise to the following theorem:

Theorem 3. Equation (4) is consistent if and only if

$$(\tilde{E}_{21}, \tilde{E}_{22}) V_B^T = U_{A2}^T E = 0 \qquad (24a)$$

and

$$U_A \begin{pmatrix} \tilde{E}_{12} \\ \tilde{E}_{22} \end{pmatrix} = E V_{B2} = 0 , \qquad (24b)$$

where $U_A = (U_{A1}, U_{A2})$ and $V_B = (V_{B1}, V_{B2})$.

The solution of equation (4), if condition (24) is satisfied, can be expressed as

$$X = V_A \begin{pmatrix} \Sigma_A^{-1} \tilde{E}_{11} \Sigma_B^{-1} & Z_1 \\ Z_2 & Z_3 \end{pmatrix} U_B^T$$

$$= A^+ E B^+ + V_A \begin{pmatrix} 0 & Z_1 \\ Z_2 & Z_3 \end{pmatrix} U_B^T , \qquad (25)$$

with the matrices Z_1, Z_2 and Z_3 being arbitrary.

Condition (24) is equivalent to those derived in [2] [5] for the consistency of equation (4).

It is easy to see that the least-squares solution of equation (4) is possible from equation (23) or (25), as the transformations in equation (23) are all orthogonal. The choice of $Z_i = 0$ in equation (25) will provide a minimum norm solution, X , and the residual of an inconsistent equation will be

$$\begin{bmatrix} 0 & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{bmatrix} .$$

The solution process in equation (23) or (25) will be numerically unstable if any SV of the matrix A or B is small. Again, it may be necessary to reset small SV to zero and transfer the resulting errors to the RHS of equation (3) to be treated as residuals.

Similar to equation (21), the QR decomposition (or even the Gaussian elimination process with suitable pivoting) can be applied to solve equation (3) - instead of the more expensive SVD in equation (23), if the rank determinations of matrices A and B are trouble-free.

6. $(AXB, FXG) = (E, H)$.

Observe in equation (2) that the matrices A and F (B^T and G^T) have the same number of columns, and apply the GSVD to the respective matrix-pairs. Equation (2) is then equivalent to:

$$\begin{cases} U_1 \Sigma_A X_1 \cdot X \cdot X_2^T \Sigma_B^T U_2^T = E , & (26a) \\ V_1 \Sigma_F X_1 \cdot X \cdot X_2^T \Sigma_G^T V_2^T = H , & (26b) \end{cases}$$

where the matrices U_i and V_i are orthogonal, and X_i are non-singular, as in equations (7) to (9).

Define $\tilde{X} = X_1 \cdot X \cdot X_2^T$, $\tilde{E} = U_1^T E U_2$ and $\tilde{H} = V_1^T H V_2$, equation (27) is equivalent to

$$\begin{cases} \Sigma_A \tilde{X} \Sigma_B^T = \tilde{E} , \\ \Sigma_F \tilde{X} \Sigma_G^T = \tilde{H} , \end{cases}$$

$$\Leftrightarrow \left(\begin{array}{ccc:c} I_A & & & \\ & S_A & & 0 \\ & & 0_A & \end{array} \right) \tilde{X} \left(\begin{array}{ccc} I_B & & \\ & S_B & \\ & & 0_B^T \\ \hdashline & 0 & \end{array} \right) = \tilde{E} , \qquad (27a)$$

and

$$\left(\begin{array}{ccc:c} 0_F & & & \\ & S_F & & 0 \\ & & I_F & \end{array} \right) \tilde{X} \left(\begin{array}{ccc} 0_G^T & & \\ & S_G & \\ & & I_G \\ \hdashline & 0 & \end{array} \right) = \tilde{H} . \qquad (27b)$$

Partitioning the matrices $\tilde{X}$, $\tilde{E}$ and $\tilde{H}$ in accordance with that of the Σ's , equation (27) leads to:

Theorem 4. Equation (2) is consistent if and only if:-

(i) $\tilde{E}_{31}, \tilde{E}_{32}, \tilde{E}_{33}, \tilde{E}_{23}, \tilde{E}_{13} = 0$; (28a)

(ii) $\tilde{H}_{13}, \tilde{H}_{12}, \tilde{H}_{11}, \tilde{H}_{21}, \tilde{H}_{31} = 0$; (28b)

(iii) $\Delta = S_A^{-1}\tilde{E}_{22}S_B^{-1} = S_F^{-1}\tilde{H}_{22}S_G^{-1}$ (29)

The solution of equation (2) can be expressed as

$$X = X_1^{-1}\tilde{X}X_2^{-T} \tag{30}$$

where

$$\tilde{X} = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12}S_B^{-1} & Z_1 & Z_2 \\ S_A^{-1}\tilde{E}_{21} & \Delta & S_F^{-1}\tilde{H}_{23} & Z_3 \\ Z_4 & \tilde{H}_{32}S_G^{-1} & \tilde{H}_{33} & Z_5 \\ Z_6 & Z_7 & Z_8 & Z_9 \end{pmatrix}, \tag{31}$$

with the matrices Z_1 to Z_9 being arbitrary.

(Proof): Conditions (28a) and (28b) are the consistency conditions of the individual equations in (27a) and (27b) respectively. (They are similar to condition (24) in Theorem 3.)

Condition (29) is the result of matching the solutions of the equations involving $\tilde{X}_{22}$ in equation (27).

Equation (31) is a trivial consequence from (27).

■

The algorithm for the solution of equation (2), (suggested by equation (27) or (31)) will be numerically unstable if any of the GSV is small, or if the matrix X_1 or X_2 is ill-conditioned. (Recall the discussion on conditions of these matrices after equation (11).)

From equation (26), the solution in (31) minimizes the 2- or F-norm of the residual of an inconsistent equation (2), but will not provide a minimum norm solution X, as X_1 and X_2 are not orthogonal.

Note that special cases of equation (2), e.g.

$$\begin{cases} AX = E , \\ XG = H , \end{cases}$$

can be similarly treated using SVD.

7. Duality.

It is not widely appreciated, but can be easily proved, that the adjoint equation of (1) is in the form of (2), and vice versa.

Starting from equation (1) and using the Kronecker product, one has

$$AXB + CYD = E$$
$$\Leftrightarrow \quad (A \otimes B^T, C \otimes D^T) \cdot v\begin{pmatrix} X \\ Y \end{pmatrix} = v(E) , \qquad (32)$$

where the column vector $v(M)$ is formed by lining up successive rows of the matrix M and transposing.

Consider the adjoint equation of equation (32), one has

$$v(Z_1^T)^T \cdot (A \otimes B^T, C \otimes D^T) = v(Z_2^T)^T$$

$$\Leftrightarrow \quad \begin{pmatrix} A^T \otimes B \\ C^T \otimes D \end{pmatrix} \cdot v(Z_1^T) = v(Z_2^T)$$

$$\Leftrightarrow \quad \begin{cases} A^T Z_1^T B^T = Z_{21}^T , \\ C^T Z_1^T D^T = Z_{22}^T , \end{cases} \text{ where } Z_2^T = \begin{pmatrix} Z_{21}^T \\ Z_{22}^T \end{pmatrix} ;$$

$$\Leftrightarrow \quad \begin{cases} BZ_1A = Z_{21} , \\ DZ_1C = Z_{22} . \end{cases} \qquad (33)$$

From equations (32) and (33), one can derive the following conditions for consistency of equations (1) and (2), using the duality property:

<u>Theorem 5</u>. Equation (1) is consistent if and only if:-

$$BZA = 0 \text{ and } DZC = 0 \Rightarrow \text{trace } (EZ) = 0 .$$

Equation (2) is consistent if and only if:-

$$BZ_1A + GZ_2F = 0 \Rightarrow \text{trace } (EZ_1+HZ_2) = 0 .$$

(Proof): From equations (32) and (33), (1) is consistent iff

$$BZA = 0 , \quad DZC = 0 \Rightarrow V(Z^T)^T \cdot V(E) = 0$$

and $V(Z^T)^T \cdot V(E) = \text{trace } (EZ)$.

Similarly, starting from equation (2), the second part of the theorem may be proved.

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Obviously, Theorem 5 holds for special cases of equations (1) and (2) (e.g. (3) and (4)) in their respective simplified forms.

8. Conclusions.

In this paper, we apply GSVD to investigate the solution of the linear matrix equations (1) and (2). Special cases in (3) and (4) are treated using SVD. Consistency conditions are derived and solutions for consistent equations are characterized. Possibilities of solving the equations in the least squares sense have also been discussed when appropriate. Additional conditions for the consistency of the equations are then derived, using the duality of equations of the form (1) and (2).

Although the paper is essentially a theoretical one, numerical algorithms for the solution of the equations are suggested and numerical considerations have always been kept in mind. More work - expecially numerical experimentations - have to be done.

Finally, the potentially very powerful tool, the GSVD, has been around for nearly ten years but applications have been surprisingly limited, compared to that of the SVD. This paper represents a moderate attempt to redress the situation.

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