DEPARTMENT OF MATHEMATICS

ON STABILITY OF DESCRIPTOR SYSTEMS

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ON STABILITY OF DESCRIPTOR SYSTEMS

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Abstract. The concept of "distance to instability" of a system matrix is generalized to system *pencils* which arise in descriptor (semi-state) systems. Difficulties arise in the case of singular systems, because the pencil can be made unstable by an infinitesimal perturbation. It is necessary to measure the distance subject to restricted, or structured, perturbations. In this paper a suitable measure for the stability radius of a generalized state-space system is defined, and a computable expression for the distance to instability is derived for regular pencils of index less than or equal to one.

1. Introduction. The concept of "distance to instability" or "stability radius" of a multivariable linear system in state-space is closely related to the "margin of stability" of such a system in the frequency domain. Measures of stability radius have recently been investigated in a number of papers and numerical methods for computing the distance to instability have been developed [1, 6, 7, 9]. In this paper we extend the concept of distance to instability to system *pencils* which arise in descriptor or generalized state-space systems described by implicit differential-algebraic equations. In Section 2 the distance measure is defined and notation is presented. A computable expression for the distance is derived in Section 3 for regular pencils of index less than or equal to one. In Section 4 different classes of perturbations are discussed and conclusions are given. Details of proofs and furthers examples are given in [2].

2. Distance to Instability — Definitions and Notation. We consider the linear time-invariant system

$$E\dot{x} = Ax + Bu \tag{1}$$

where E, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$, $\operatorname{rank}[B] = p$, and $q \equiv \operatorname{rank}[E] \leq n$. The system (1) is said to be *solvable* if and only if there exists a unique solution for any given sufficiently differentiable control function $u(t)$ and any given admissible initial conditions corresponding to a given $u(t)$ [3, 10]. It has been shown [3] that system (1) is *solvable* if and only if the system pencil $(\alpha A - \beta E)$ is *regular*, that is $\det(\alpha A - \beta E) \neq 0$ for some

$$(\alpha,\beta) \in \mathbf{C} \times \mathbf{C}\backslash\{0,0\}. \tag{2}$$

For a regular system pencil, the solutions to (1) can be characterized in terms of the eigenstructure of the pencil. The generalized eigenvalues are defined by the pairs $(\alpha_j,\beta_j) \in \mathbf{C} \times \mathbf{C}\backslash\{(0,0)\}$ such that

$$\det(\alpha_j A - \beta_j E) = 0, \qquad j = 1, 2, 3, \ldots, n. \tag{3}$$

Without loss of generality, it can be assumed that $|\alpha_j|^2 + |\beta_j|^2 = 1$. If $\alpha_j \neq 0$, then $\lambda_j = \beta_j/\alpha_j$ is a *finite* eigenvalue and if $\alpha_j = 0$, then $\lambda_j \sim \infty$ is an *infinite* eigenvalue of the system. The right and left generalized eigenvectors and principal vectors are given by the columns of the non-singular matrices $[X_r, X_\infty]$ and $[Y_r, Y_\infty]$ (respectively) which transform the pencil into the Kronecker Canonical Form (KCF)

$$(4) \qquad Y^T AX = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix}, \qquad Y^T EX = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix},$$

where J is the r-by-r Jordan matrix associated with the $r \leq q$ finite eigenvalues of the pencil, and N is a nilpotent matrix, also in Jordan form, corresponding to the $n - r$ infinite eigenvalues [4]. The degree of nilpotency $m \in \mathbf{Z}$ such that $N^m = 0$, $N^{m-1} \neq 0$, is called the *index* of the system.

A simple example of a regular index one system is given by the differential-algebraic equations

$$(5) \qquad \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,$$

where E_{11} and A_{22} are square and of full rank. The first block of equations describes the dynamic behavior of the system, while the second set give algebraic constraints on the states. Such systems arise, for example, where path constraints are imposed on the dynamic response.

For a regular system, the solution to (1) can be given explicitly in terms of the KCF [3, 10]. It is easily seen that with $u \equiv 0$ $\forall t$, the response $x(t)$ of the system converges to a position of stable equilibrium at the origin, i.e., $x(t) \to 0$ as $t \to \infty$ for any admissible $x(0)$, if and only if the finite eigenvalues of the system all lie in the left half of the complex plane. We make the following definition.

DEFINITION 1. *If the pencil $(\alpha A - \beta B)$ is regular and its $r \leq q$ finite eigenvalues $\lambda_j = \beta_j/\alpha_j$ satisfy* $\mathrm{Re}(\lambda_j) < 0$, *$j = 1, 2, 3, \ldots, r$, then the pencil is* stable. *Otherwise it is* unstable. □

For a *standard* stable system (with $E = I$), the distance to instability, or radius of stability, is measured in terms of the minimum perturbation δA to the matrix A required to make the perturbed system unstable [1, 6, 7, 9]. For descriptor systems this definition is not immediately applicable. If we consider perturbations $(\delta A, \delta E)$ to the system pair (A, E), it is easy to see from (4) that an infinitesimal perturbation to the nilpotent part of the pencil can change its eigenstructure and the solution space of the system.

To illustrate this consider the system (of type (5))

$$(6) \qquad \begin{aligned} \dot{x}_1 &= -2x_1 + x_3 \\ \dot{x}_2 &= -x_2 + x_4 \\ \dot{x}_3 &= -u_1 \\ \dot{x}_4 &= -u_2 \end{aligned}$$

with system matrices

$$(7) \qquad A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The system (6) is regular, index one and stable with two finite eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$. If we introduce the perturbations $(\delta A, \delta E)$, where

$$(8) \qquad \delta A = 0, \qquad \delta E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_1 & \epsilon_2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\epsilon_1 > 0$, $\epsilon_2 = 0$, then the perturbed pencil $(\alpha(A + \delta A) - \beta(E + \delta E))$ is still regular and index one but it has *three* finite eigenvalues, $\lambda_1 = -2$, $\lambda_2 = -1$ at $\lambda_3 = 1/\epsilon_1 > 0$ and is clearly unstable for any positive value of ϵ_1. The solution of the perturbed system has more degrees of freedom than the original system, and the admissible initial conditions are altered. If the same perturbation is introduced, but we let $\epsilon_1 = 0$ and $\epsilon_2 > 0$, then the perturbed pencil remains regular, but has index equal to two. The solution space of the system is altered and the admissible controls must be smoother. In both cases the perturbation causes the algebraic constraint to become differential.

If we exclude perturbations which alter the nilpotent part of the pencil, then the finite eigenvalues of the perturbed pencil depend continuously on $(\delta A, \delta E)$ and the "distance to instability" of the pencil can be measured in terms of the minimum perturbation required for a finite eigenvalue to move to the imaginary axis (compactified by adding a point at infinity), or for the pencil to lose regularity. In practice, it is reasonable to allow only perturbation such that the system remains solvable for the same fixed class of admissible controls and initial conditions. This is ensured if the nilpotent structure of the pencil is preserved, or more specifically in the KCF (4), the nilpotent Jordan matrix N and the corresponding left invariant space spanned by the rows of Y_∞^T are both preserved under the allowable perturbations.

For systems of the type (5) such restrictions exclude perturbations which cause the algebraic constraints to become differential, which is a natural limitation. Allowable perturbations can, nevertheless lead to systems which are unstable, different index or not regular. Such perturbations are finite and measurable, however. Examples of such perturbations are given in [2].

As a brief illustration, we consider the system (6) subject to perturbations $(\delta A, \delta E)$ where

$$(9) \qquad \delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \delta E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_1 & \epsilon_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For small values of the parameters τ, ϵ_1, ϵ_2, these perturbations do not affect the nilpotent structure of the system. For larger values of the parameters, however, the perturbations can alter the nilpotency of the pencil or cause it to become unstable or to lose regularity. If, for instance, we select $\epsilon_1 = \epsilon_2 = 0$, then in the limit as $\tau \to 1$, a finite eigenvalue moves to the imaginary axis and the system becomes unstable. If $\tau = \epsilon_2 = 0$ then as $\epsilon_1 \to -1$, a finite eigenvalue becomes infinite and , in the limit, the nilpotent structure is changed, although the system is still of index one. Similarly, if $\tau = 0$ and $\epsilon_2 = -\epsilon_1$, then as $\epsilon_1 \to -1$ a finite eigenvalue becomes infinite, but in this case, the index of the system is increased to two. Finally, if $\epsilon_2 = 0$ and $\epsilon_1 = -\tau$, then in the limit as $\tau \to 1$, the system loses regularity.

Motivated by these examples we define the distance $\rho(A, E)$ from the pencil $(\alpha A - \beta E)$ to the "nearest" unstable pencil, as the minimum perturbation which causes the pencil to become unstable, to change its nilpotent structure or to lose regularity, measured over a class $\mathcal{D}(A, E)$ of allowable perturbations. To make the definition more precise we introduce the following notation. We denote the pencil $\alpha A - \beta B)$ by (A, B) and the set of unstable (complex) pencils by $\mathcal{U}_n$; that is

$$(10) \quad \mathcal{U}_n \equiv \{(A, E) \mid A, E \in \mathbf{C}^{n \times n}, (A, E) \text{ is regular, and } \det(\alpha A - \beta E) = 0 \text{ for some } \alpha, \beta \in \mathbf{C} \text{ with } \alpha \neq 0, \mathrm{Re}(\beta/\alpha) \geq 0 \}$$

We denote the nilpotent structure of the pencil (A, E) by $\mathrm{nil}(A, E)$, where the nilpotent structure specifically refers to the nilpotent Jordan matrix N of the KCF (4) and to the corresponding left invariant space spanned by the rows of Y_∞^T. The set of allowable perturbations is then defined by

$$(11) \quad \mathcal{D} = \{(\delta A, \delta E) \mid \delta A, \delta E \in \mathbf{C}^{n \times n} \text{ and } \forall t \in [0,1], (A + t\delta A, E + t\delta E) \text{ is regular and } \mathrm{nil}(A + t\delta A, E + t\delta E) = \mathrm{nil}(A, E)\}$$

We observe that if $(\delta A, \delta E)$ belongs to the *closure* of $\mathcal{D}(A, E)$ but not to the set itself, then the perturbation alters the nilpotency or the regularity of the pencil. We now define the measure of distance to instability as follows.

DEFINITION 2. *The* distance to instability *or* radius of stability *of the stable regular pencil* (A, E) *is given by*

$$\rho(A, E) \equiv \min_{(\delta A, \delta E) \in \bar{\mathcal{D}}(A,E)} \{\|[\delta A|\delta E]\|_F \mid (A + \delta A, E + \delta E) \in \mathcal{U}_n \text{ or } (\delta A, \delta E) \notin \mathcal{D}(A, E)\},$$

where $\|\cdot\|_F$ *denotes the Frobenius norm and* $\bar{\mathcal{D}}(A, E)$ *denotes the closure of the set of allowable perturbations* $\mathcal{D}(A, E)$ *defined by (11).* □

From Definition 2 it follows immediately that if $(\delta A, \delta E) \in \mathcal{D}(A, E)$ and $\|[\delta A|\delta E]\|_F < \rho(A, E)$, then $(A + \delta A, E + \delta E)$ is *stable*. We remark that $\rho(A, E)$ measures that distance to the nearest *complex* pencil which is unstable, has a different nilpotent structure or is not regular. In practice we may be interested only in *real* perturbation to the pencil. The measure $\rho(A, E)$ gives a lower bound for this case. In the next section we derive a computable expression for the distance to instability $\rho(A, E)$ for systems which are of index at most one.

3. Regular Index One Systems. We now assume that (A, E) is a regular pencil of index less than or equal to one. Then (A, E) has precisely $q \equiv \mathrm{rank}[E]$ finite eigenvalues and the nilpotent structure of the systems pencil is given by $N = 0$ and $\mathcal{R}\{Y_\infty^T\} = \mathcal{N}_L\{E\}$, where $\mathcal{R}\{\cdot\}$ and $\mathcal{N}\{\cdot\}$ denote the range and left null spaces respectively.

In order to derive a computable measure for the radius of stability we must obtain an explicit description of the set $\mathcal{D}(A, E)$ of allowable perturbations. We use the following result form [8]:

LEMMA 3.1. *The pencil* (A, E) *is regular, index less than or equal to one if and only if*

$$(12) \quad \mathrm{rank}\begin{bmatrix} Y_\infty^T A \\ E \end{bmatrix} = n$$

where $\mathcal{R}(Y_\infty^T) = \mathcal{N}_L(E)$. From this result we can show the following (See [2] for a proof):

LEMMA 3.2. *If (A,E) is regular, index less than or equal to one, then the set* $\mathcal{D}(A,E)$ *is equivalent to the set of all complex perturbations* $(\delta A, \delta E)$ *satisfying*

(i) $\text{rank}[E + \delta E] = \text{rank}[E]$,

(ii) $Y_\infty^T \delta E = 0$,

(iii) $\text{rank} \begin{bmatrix} Y_\infty^T (A + \delta A) \\ E + \delta E \end{bmatrix} = n$, *where* $\mathcal{R}(Y_\infty^T) = \mathcal{N}_L(E)$.

□

We can show furthermore that the measure $\rho(A,E)$ is invariant under orthogonal transformations of the pencil, and therefore, it is sufficient to compute $\rho(A,E)$ only for a certain class of pencils. We have the following lemma (See [2] for proof).

LEMMA 3.3. *If* (A,E) *is regular, index less than or equal to one, then there exist orthogonal matrices* P, Q *such that*

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad PEQ = \begin{bmatrix} E_{11} & E_{22} \\ 0 & 0 \end{bmatrix} \tag{13}$$

where $\text{rank}\, E_{11} = \text{rank}\, E \equiv q$ *and* $\text{rank}\, A_{22} = n - q$ *and*

$$\rho(A,E) = \rho\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} E_{11} & E_{22} \\ 0 & 0 \end{bmatrix} \right).$$

Furthermore, $(\delta A, \delta E) \in \mathcal{D}(A,E)$ *implies*

$$P\delta AQ = \begin{bmatrix} \delta A_{11} & \delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix} \qquad P\delta EQ = \begin{bmatrix} \delta E_{11} & \delta E_{22} \\ 0 & 0 \end{bmatrix}. \tag{14}$$

□

In order to evaluate $\rho(A,E)$ we may assume without loss of generality, that (A,E) is already in the partitioned form (13). We define

$$H(\theta,\omega) = \begin{bmatrix} \theta A_{11} & \theta A_{12} \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} i\omega E_{11} & i\omega E_{22} \\ 0 & 0 \end{bmatrix}. \tag{15}$$

We can now show that perturbations to $H(\theta,\omega)$ of form

$$\Delta(\theta,\omega) = \begin{bmatrix} \theta\delta A_{11} & \theta\delta A_{12} \\ \delta A_{21} & \delta A_{22} \end{bmatrix} - \begin{bmatrix} i\omega\delta E_{11} & i\omega\delta E_{12} \\ 0 & 0 \end{bmatrix} \tag{16}$$

can cause the perturbed matrix to become *singular* if and only if the allowable perturbations $\delta A, \delta E)$ (in partitioned form (14)) fail to satisfy the conditions of Lemma 3.2 or else cause the pencil (A,E) to become unstable. We can thus establish the following (See [2].) for details.

LEMMA 3.4. *If* (A,E) *is regular, index less than or equal to one, then*

$$\rho(A,E) = \inf_{\substack{\theta,\,\omega \in \mathbf{R} \\ \theta^2 + \omega^2 = 1}} \{\|[\delta A|\delta E]\|_F \,|\, H(\theta,\omega) \text{ is singular}\}$$

where $H(\theta,\omega)$ *and* $\Delta(\theta,\omega)$ *are given by (15) and (16) respectively.* □

Finally we can show that an equivalence exists between a perturbation Δ which makes H singular and a perturbation $(\delta A, \delta E) \in \mathcal{D}(A,E)$ such that $\|[\delta A|\delta E]\|_F = \|\Delta\|_F$. This gives us the main theorem.

THEOREM 3.5. *If (A,E) is regular, index less than or equal to one, then*

$$\rho(A,E) = \inf_{\substack{\theta,\,\omega \in \mathbf{R} \\ \theta^2+\omega^2=1}} \sigma_{\min}\{H(\theta,\omega)\}, \tag{17}$$

where $\sigma_{\min}\{\cdot\}$ denotes the smallest singular value, and $H(\theta,\omega)$ is given by (15). □

The proof of the theorem is given in [2] and depends on the well-known result [5] that for a nonsingular matrix H, the perturbed matrix $H+\Delta$ is singular only if $\|\Delta\|_F \geq \sigma_{\min}\{H\}$, with equality of some Δ.

We remark that the measure (17) is computable, and we are now developing reliable and efficient numerical algorithms for evaluating this measure, based on the bisection methods derived by [1] and [6] for the standard problem.

A simple, but expensive, method for computing $\rho(A,E)$ is to apply a standard software library program to minimize the one parameter function

$$f(\alpha) \equiv \sigma_{\min}\{H(\cos(\alpha), \sin(\alpha))\}. \tag{18}$$

To illustrate the results of Theorem 3.5, we apply this technique to determine the radius of stability for the system of example (6). We find that $\rho(A,E) = .6180$, and the minimizing perturbation given to four figures by

$$\delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -.2764 & 0 & -.1708 \\ 0 & .4472 & 0 & .2764 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \delta E = 0$$

causes the pencil to become unstable with an eigenvalue at the origin.

4. Conclusions. In the previous sections we investigate the distance of a matrix pencil to the "nearest" unstable pencil. We demonstrate that a measure of this distance in terms of perturbations of the pencil is nil unless the allowable perturbations are restricted. A natural set of restrictions is defined and the measure of distance to instability is established. For regular systems of index less than or equal to one, the restrictions simply imply that in the differential-algebraic equations the perturbations cannot cause the algebraic part of the system to become differential. That such perturbations do not arise in practice is frequently ensured by underlying physical aspects of the system. A computable expression for the distance to instability of a regular system of index less than or equal to one is also derived and illustrative examples are given.

We observe that for some systems other restrictions on the perturbations might be appropriate. For instance for systems arising in the form

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{19}$$

it might be natural to assume also that allowable perturbations keep the right null space of E invariant, i.e., $\delta E X_\infty = 0$. Alternatively it might be natural to assume that the algebraic and differential state variables of the system remain decoupled, i.e., $\delta A_{21} = 0$, so that the solution space of the homogeneous system is not altered. The results of the previous sections can also be extended to obtain computable measures of the distance to instability over such sets of allowable perturbations. Detailed descriptions of these generalizations are given in [2].

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