MOVING FINITE ELEMENTS FOR NONLINEAR DIFFUSION PROBLEMS IN ONE AND TWO DIMENSIONS

by

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ABSTRACT

The Moving Finite Element (MFE) method is applied to nonlinear diffusion equations of the form

$$u_t = \underline{\nabla} \cdot (u^n \underline{\nabla} u) \qquad (n > 0)$$

whose solutions exhibit steep moving fronts. After a crucial change of dependent variable we show how to derive semi-discrete MFE equations for the position of the moving interface. We show that these equations are exact in the one-dimensional case and also derive a maximum principle for the gradient of the MFE approximation in this case. No regularization terms are needed and explicit time stepping, using an adaptive time step based on the size of the diffusion coefficient, is found to be sufficient to give a good solution in time.

1. INTRODUCTION

The MFE method for time dependent partial differential equations introduced by Miller & Miller [7] has been used with considerable success for parabolic problems (see Miller [8], Miller & Miller [7], Gelinas, Doss & Miller [3], Herbst [4]). The method has also been shown to be particularly appropriate for hyperbolic problems by Wathen [14] and the structure of the MFE matrix has been analysed by Wathen & Baines [15]. This analysis has led to a very efficient explicit solution procedure avoiding the problem of additional regularization terms as used by earlier authors and indeed in one dimension the method has been shown to be a purely local method (see Baines [2]).

The approach introduced by Wathen & Baines has been applied to convection-diffusion problems in one dimension (see Johnson [5]) and in this report the method is extended to solve nonlinear diffusion problems.

In Section 2 we introduce the class of nonlinear diffusion problems

$$u_t = \underline{\nabla} \cdot (u^n \underline{\nabla} u) \qquad (n > 0) \tag{1.1}$$

and derive similarity solutions in q space dimensions, assuming radial symmetry.

In Section 3 we discuss a change of dependent variable and show that one particular transformation is very appropriate for this class of problems and also very well suited to the application of MFE.

In Section 4 the MFE method is introduced in one and two dimensions and application of the method to the class of problems described in Section 2 is discussed. Applying the MFE method to the transformed variable using the transformation described in Section 3 allows us to derive semi-discrete equations for the position of the moving interface which are exact in one dimension.

In Section 6 we derive a maximum principle for the gradient of the MFE approximation in one dimension using the elementwise formulation (c.f. Baines [2]).

We describe in Section 7 a time-stepping procedure which prevents element folding without the addition of regularization terms, and also give a simple adaptive time step algorithm.

In Section 8 numerical results are compared with similarity solutions in one and two dimensions for various values of the parameter n in the partial differential equation.

Section 9 gives a conclusion from the analysis and numerical results of the previous sections.

2. NONLINEAR DIFFUSION EQUATIONS

We consider nonlinear diffusion equations of the form

$$u_t = \underline{\nabla} \cdot (u^n \underline{\nabla} u) \qquad (n > 0) \tag{2.1}$$

whose solutions exhibit steep moving fronts. Physical examples of processes governed by such equations are bursts of radiation, seepage of liquids into porous media and the spreading of a thin liquid film under gravity. Similarity solutions exist for many such problems and have been studied by many authors, e.g. Ames [1], Tayler, Ockendon & Lacey [12] and Zel'dovich & Kompaneets [16]. Finite difference solutions for such problems in one dimension are discussed in Richtmyer & Morton [11] and have been studied among others by Meek & Norbury [6] and Tomoeda [13].

Similarity Solution

A similarity solution to equation (2.1) exists in q space dimensions assuming radial symmetry and constant total thermal energy (see Ames [1]).

Assuming spherical symmetry in q space dimensions (2.1) reduces to the one dimensional equation

$$r^{1-q} \frac{\partial}{\partial r} \left(r^{q-1} u^n \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial t} \quad . \tag{2.2}$$

We assume the form of the similarity variable is $X = r/R(t)$ and look for a trial solution of the form

$$u(r,t) = V(t)Y(r/R(t)) \quad . \tag{2.3}$$

Substituting (2.3) into (2.2) yields

$$V^{n+1}(t) X^{1-q} \frac{d}{dX} \left[X^{q-1} Y^n Y' \right] = R(t)[RYV' - XVY'R] \quad . \tag{2.4}$$

The RHS of (2.4) is separable iff the condition

$$VR' = - AV'R$$

is satisfied with A constant so that

$$R(t) = [V(t)]^{-A} \quad . \tag{2.5}$$

The RHS of (2.4) takes the form

$$R^2V(Y + AXY')$$

so that (2.4) separates into

$$\frac{X^{1-q}\frac{d}{dX}(X^{q-1}Y^nY')}{Y + AXY} = \frac{R^2V'}{V^{n+1}} = -B \tag{2.6}$$

with B constant.

We now consider the case of constant total thermal energy, corresponding to the diffusion of a fixed quantity of heat initially at the origin, i.e.

$$E = s(q)\int_0^\infty u(r,t)r^{q-1}dr = \text{constant} \tag{2.7}$$

where $s(q)$ is the area of the unit sphere in q dimensions.

Substituting (2.3) into (2.7) gives

$$E = R^q(t)V(t)s(q)\int_0^1 Y(X)X^{q-1}dX \quad .$$

Hence we require that

$$R^q(t)V(t) = V^{1-Aq}$$

be constant. Hence

$$A = 1/q$$

and

$$R = V^{-1/q} \tag{2.8}$$

Now in (2.6)

$$\frac{R^2V'}{V^{n+1}} = -B$$

and substituting for R from (2.8) yields

$$\frac{V^{-2/q}V'}{V^{n+1}} = -B$$

$$\Rightarrow \quad V^{-(\frac{2}{q} + n + 1)} V' = -B$$

$$\Rightarrow \quad V = [B(n + \frac{2}{q})t]^{\frac{-q}{nq+2}} , \tag{2.9}$$

so that

$$R(t) = V^{-1/q} = [B(n + \frac{q}{2})t]^{\frac{1}{nq+2}} . \tag{2.10}$$

With $A = 1/q$ the spatial part of (2.6) becomes

$$\frac{d}{dx}\left[X^{q-1}Y^nY'\right] = -\frac{B}{q}\frac{d}{dX}[X^qY] \tag{2.11}$$

which integrates once to give

$$X^{q-1}Y^nY' = -\frac{B}{q}X^qY + \alpha .$$

If $Y(0) = C$ and $X^{q-1}Y' \to 0$ as $X \to 0$ then $\alpha = 0$ and a second integration gives

$$Y^n = -\frac{nBX^2}{2q} + \gamma .$$

Hence

$$Y = \gamma^{1/n}\left(1 - \frac{nBX^2}{2\gamma q}\right)^{\frac{1}{n}}$$

$$Y(0) = C \Rightarrow \gamma = C^n \qquad \text{and if we set} \quad nB = 2qC^n$$

we have

$$Y(X) = C(1 - X^2)^{\frac{1}{n}} \tag{2.12}$$

where the constant C may be evaluated from condition (2.7) which requires that

$$E = s(q)V(t)\int_0^1 Y(X)X^{q-1}R^{q-1}RdX$$

be constant.

From (2.8) we have $V(t)R^q = 1$ and hence

$$E = s(q)\int_0^1 Y(X)X^{q-1}dX \quad . \tag{2.13}$$

Substituting (2.12) into (2.13) gives

$$E = s(q)\ C\int_0^1 (1 - X^2)^{1/n}X^{q-1}dX$$

$$= s(q)\ \frac{C}{2}\ B(1/n + 1,\ q/2)$$

$$\Rightarrow C = \frac{2E}{s(q)}\ (1 + \frac{nq}{2})\ \frac{\Gamma(1/n + q/2)}{\Gamma(1/n)\Gamma(q/2)} \quad . \tag{2.14}$$

Using $nB = 2qC^n$ and substituting in (2.10) gives

$$R(t) = \left[C^n\ \frac{2}{n}\ (nq + 2)t\right]^{\frac{1}{nq+2}} \tag{2.15}$$

and the similarity solution (2.3) may now be expressed as

$$u(r,t) = \begin{cases} \dfrac{C}{R^q}\left(1 - \left(\dfrac{r}{R}\right)^2\right)^{\frac{1}{n}} & 0 \leq r \leq R(t) \\ 0 & r > R(t) \end{cases} \tag{2.16}$$

where $R(t)$ is defined by (2.15) and the constant C by (2.14).
(N.B. (2.14) differs from the result in Ames [1] which is incorrect).

For the numerical results given in Section 8 we have taken the value of the constant E in (2.14) to be unity.

3. CHANGE OF DEPENDENT VARIABLE

A finite difference solution of equation (2.1) is given by Tomoeda [13] who suggests that a change of dependent variable of the form $v = u^n$ is appropriate for such problems. Similar problems are at present being studied at Reading by Please & Sweby [10] who suggest that for equations of the form

$$u_t = \underline{\nabla} \cdot (D(u)\underline{\nabla}u)$$

whose solutions exhibit steep moving fronts an appropriate change of dependent variable is given by

$$\phi = -\int \frac{D(u)}{u}\, du \tag{3.1}$$

which yields the equation

$$\phi_t = D(u)\underline{\nabla}^2\phi - (\underline{\nabla}\phi)^2 \quad .$$

If $\underset{\sim}{w}$ is the velocity of the moving wave and $\underset{\sim}{F}$ the flux defined by

$$\underset{\sim}{F} = -D(u)\underline{\nabla}u = u\underset{\sim}{w}$$

then

$$\underset{\sim}{w} = \frac{-D(u)\underline{\nabla}u}{u}$$

$$= \underline{\nabla}\left(-\int \frac{D(u)}{u}\, du\right) \tag{3.2}$$

If also we define a velocity potential ψ, such that

$$\underset{\sim}{w} = \nabla\psi$$

then from (3.2) we see that

$$\psi = -\int \frac{D(u)}{u}\, du \quad .$$

Referring back to (3.1) we see that the new variable ϕ is precisely the velocity potential.

In the case of equation (2.1) we have $D(u) = u^n$ and equation (3.1) gives the new variable $\phi = -\frac{u^n}{n}$, which differs from the transformation suggested by Tomoeda only by a constant factor.

We proceed with the substitution $v = u^n$ which yields

$$v_t = v\underline{\nabla}^2 v + \frac{1}{n}(\underline{\nabla} v)^2 \quad . \tag{3.3}$$

Since we are interested in moving wave solutions of (2.1) which have compact support we must look at the behaviour of the moving interface $u = 0$ in (2.1).

If we recall the form of (2.1), that is

$$u_t = \underline{\nabla} \cdot (u^n \underline{\nabla} u) = u^n \underline{\nabla}^2 u + n u^{n-1} (\underline{\nabla} u)^2$$

then it is clear that in this form the equation gives us no information to derive the interface velocity, since if we assume that $\underline{\nabla} u$ and $\underline{\nabla}^2 u$ remain bounded as $u \to 0$ then $u_t \to 0$ as $u \to 0$. If however we consider the transformed equation (3.3) then as $v \to 0$ we have (assuming $\underline{\nabla}^2 v$ remains bounded)

$$v_t = \frac{1}{n}(\underline{\nabla} v)^2 \quad . \tag{3.4}$$

We may now proceed to solve the hyperbolic equation (3.4) to give the velocity of the moving interface. Moreover, this approach is well suited to the application of MFE since it has been shown that the method is particularly successful for hyperbolic problems (see Wathen [14]).

Indeed it may be shown that if we were to choose a transformation of the form $v = u^p$ in (2.1) then the value $p = n$ is the only value for p which yields a bounded and non-zero expression for the velocity of the interface as $u \to 0$, assuming that $\underline{\nabla} u$ and $\underline{\nabla}^2 u$ remain bounded as $u \to 0$.

4. MFE METHOD

In general we consider the solution of the time dependent partial differential equation

$$u_t = L(u) \tag{4.1}$$

where L is some nonlinear spacial differential operator. We shall derive the MFE equations in one and two dimensions as follows.

4.1 One Dimension

We seek an approximate piecewise linear MFE solution of (4.1) in the form

$$U = \sum_{1}^{N} U_j(t)\alpha_j(x, \underline{s}(t)) \tag{4.2}$$

where the parameters $U_j(t)$ are the nodal amplitudes and α_j are the standard piecewise linear finite element basis functions with an additional dependence on the vector $\underline{s}(t)$ of time dependent nodal positions.

Differentiating wrt time in (4.2) yields

$$U_t = \sum_{1}^{N} (\dot{U}_j\alpha_j + \dot{s}_j\beta_j) \tag{4.3}$$

where the parameters $\dot{s}_j$ are the nodal velocities and the functions β_j are a second type of basis function which have the same support as α_j but are discontinuous at the points s_j. The functions β_j are defined over each element by

$$\beta_j = -\frac{\partial U}{\partial x}\alpha_j$$

and a typical β_j is shown in Fig. 4.1 below.

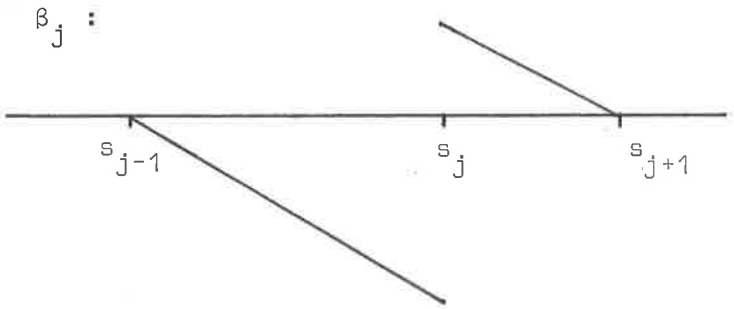

FIG. 4.1

We now proceed by minimising the residual

$$\| U_t - L(U) \|^2_{L_2}$$

with respect to the 2N parameters $\dot{U}_j$, $\dot{s}_j$ $(j = 1,\ldots,N)$ to give the 2N equations

$$\langle U_t - L(U), \begin{matrix}\alpha_i \\ \beta_i\end{matrix} \rangle = 0 \qquad (i = 1,\ldots,N)$$

which may be written as a system of 2N ordinary differential equations in the form

$$A(\underline{y})\dot{\underline{y}} = \underline{g}(\underline{y}) \tag{4.4}$$

where $\underline{y} = (U_1, s_1, \ldots, U_N, s_N)^T$. Equations (4.4) are called the MFE equations and the matrix A is the MFE matrix which is symmetric and block 2x2 tridiagonal with blocks given by

$$A_{ij} = \begin{bmatrix} \langle \alpha_i, \alpha_j \rangle & \langle \alpha_i, \beta_j \rangle \\ \langle \beta_i, \alpha_j \rangle & \langle \beta_i, \beta_j \rangle \end{bmatrix} .$$

The RHS vector $\underline{g}(\underline{y})$ is defined by

$$\begin{aligned} g_{2i-1} &= \langle L(U), \alpha_i \rangle \\ g_{2i} &= \langle L(U), \beta_i \rangle \end{aligned} \qquad (i = 1,\ldots,N) .$$

4.2 Two Dimensions

In two dimensions we seek a piecewise linear MFE solution of (4.1) on triangular elements in the form

$$U_t = \sum_1^N U_j(t)\alpha_j \, (x, y, \underline{r}(t)) \tag{4.5}$$

where the parameters $U_j(t)$ are the nodal amplitudes and the basis functions α_j are the standard piecewise linear finite element basis functions on triangles with an extra dependence on the vector $\underline{r}(t)$ of time dependent nodal positions

in the xy plane.

Differentiating with respect to time in (4.5) yields

$$U_t = \sum_1^N (\dot{U}_j \alpha_j + \dot{X}_j \beta_j + \dot{Y}_j \gamma_j) \tag{4.6}$$

where β_j and γ_j are basis functions defined over each element by

$$\beta_j = -\frac{\partial U}{\partial x} \alpha_j \qquad \gamma_j = -\frac{\partial U}{\partial y} \alpha_j$$

which have the same support as α_j but are discontinuous across element edges through the node j (see Fig. 4.2).

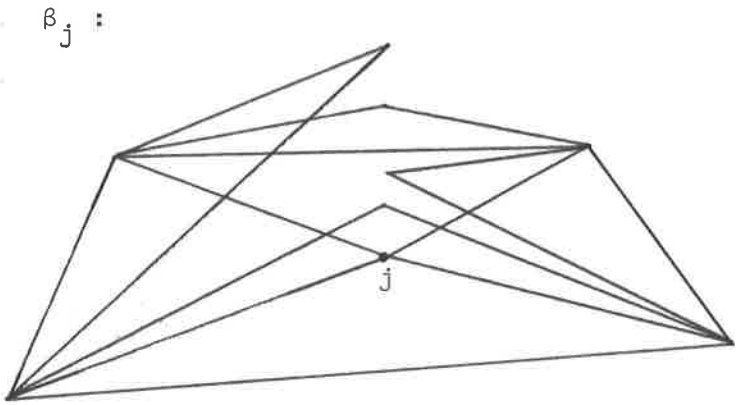

FIG. 4.2

We now proceed by minimising the residual

$$\| U_t - L(U) \|_{L_2}^2$$

with respect to the 3N parameters $\dot{U}_j$, $\dot{X}_j$, $\dot{Y}_j$ $(j = 1,\ldots,N)$ to give the MFE equations

$$A(\underline{y})\dot{\underline{y}} = \underline{g}(\underline{y}) \tag{4.7}$$

where $$\underline{y} = (U_1, X_1, Y_1, \ldots, U_N, X_N, Y_N)^T .$$

Having now derived the MFE equations in one and two dimensions we now proceed to consider the solution of these equations.

4.3 Solution of the MFE Equations

It has been shown by Wathen & Baines [15] that in one dimension a decomposition of the MFE matrix exists in the form $A = M^T C M$ where M and C are block $2x2$ diagonal matrices. Hence A may be inverted explicitly in general since $A^{-1} = M^{-1} C^{-1} M^{-T}$, and indeed the method may be shown to be enitrely local, involving only the inversion of $2x2$ matrices (see Baines[2]).

In higher dimensions this explicit inversion property is lost but it has been shown by Wathen [14] that, if D is the matrix of diagonal blocks of A, the eigenvalue spectrum of $D^{-1}A$ always satisfies

$$\rho(D^{-1}A) \in [\tfrac{1}{2},\ 1 + d/2]$$

where d is the number of space dimensions. This result is independent of the number of unknowns and the mesh geometry. It follows that the Conjugate Gradient method with preconditioning by D^{-1} will converge very rapidly, and in two dimensions we find about 10 iterations sufficient.

Additional regularization terms were used by the original authors (see Miller & Miller [7], Miller[8]) to prevent singularity of the MFE matrix which occurs in the case of collinearity of the solution over adjacent elements (parallelism) and node overtaking (element folding in higher dimensions). These regularization terms introduced a large amount of stiffness into the MFE equations and necessitated the use of implicit stiff solvers: the original authors were then unable to exploit the fast inversion procedures described above.

Here we have introduced no regularization terms, the problem of parallelism being overcome by a constrained minimization (see Wathen [14]) and that of element folding by an explicit time-stepping procedure described in Section 7. This allows us to use the very rapid inversion of the MFE matrix at each time level as described above, and explicit Euler time stepping. For the time step t^n to t^{n+1} we then have

$$\underline{y}^{n+1} = \underline{y}^{n} + \Delta t A^{-1}(\underline{y}^{n})\underline{g}(\underline{y}^{n})$$

where $A^{-1} = M^{-1}C^{-1}M^{-T}$ in one dimension.

5. APPLICATION OF MFE TO NONLINEAR DIFFUSION EQUATIONS

Using the transformation $v = u^n$ in equation (2.1) discussed in Section 3 we seek a piecewise linear MFE solution of

$$v_t = v\underline{\nabla}^2 v + \frac{1}{n}(\underline{\nabla} v)^2 \qquad (5.1)$$

in the form

$$V = \sum_1^N V_j \alpha_j(x,y,\underline{r}(t)) \quad .$$

On the RHS of the resulting MFE equations we need to evaluate the inner products

$$< V\underline{\nabla}^2 V + \frac{1}{n}(\underline{\nabla} V)^2, \begin{matrix}\alpha_i \\ \beta_i \\ \gamma_i\end{matrix} > \qquad (i = 1,\ldots,N) \quad . \qquad (5.2)$$

Evaluation of these inner products presents an immediate problem since $\underline{\nabla}^2 V$ does not have a finite L_2 norm and moreover the basis functions β_i, γ_i are discontinuous at the nodes and across element edges.

This problem may be overcome by using δ-mollified basis functions (see Miller & Miller [7]) or by recovering a smoother function by fitting a polynomial locally to the function V or its gradient $\underline{\nabla} V$. The problem is discussed in some detail in Johnson [5] and in Baines [2]. In one dimension we use quadratic recovery of the gradient V_x as described in [5], and a generalisation of this reocvery procedure has been used successfully in two dimensions for a linear diffusion problem. However in the two dimensional non-linear case we have followed the approach used by Mueller [9] which is described below.

5.1 Mueller Inner Products

We first consider the more general problem of evaluating inner products of the form

$$< \underline{\nabla} \cdot (D\underline{\nabla} U), \begin{matrix}\alpha_i \\ \beta_i \\ \gamma_i\end{matrix} > \qquad (5.3)$$

where $D = D(U)$ is some non-linear function of the piecewise linear approximation U.

The inner products

$$\langle \underline{\nabla} \cdot (D\underline{\nabla}U), \alpha_i \rangle$$

may be evaluated using integration by parts as in the Fixed Finite Element Method, since the basis functions α_i are continuous at the nodes and across element edges.

However, for the basis functions β_i, γ_i defined over each element by

$$\beta_i = -\frac{\partial U}{\partial x}\,\alpha_i \quad , \qquad \gamma_i = -\frac{\partial U}{\partial y}\,\alpha_i \quad ,$$

we require to evaluate in (5.3)

$$\langle \underline{\nabla} \cdot (D\underline{\nabla}U), -U_x\alpha_i \rangle \quad \text{and} \quad \langle \underline{\nabla} \cdot (D\underline{\nabla}U), -U_y\alpha_i \rangle \quad .$$

Now,

$$\langle \underline{\nabla} \cdot (D\underline{\nabla}U), -U_x\alpha_i \rangle = -\int_{\Omega} \alpha_i U_x \underline{\nabla} \cdot (D\underline{\nabla}U)\, d\Omega$$

(where Ω is the patch of elements around node i consisting of the support of α_i)

$$= -\int_{\Omega} \underline{\nabla} \cdot (\alpha_i U_x D\underline{\nabla}U)\, d\Omega + \int_{\Omega} \underline{\nabla} \cdot (\alpha_i U_x) \cdot D\underline{\nabla}U \, d\Omega$$

Using Green's Theorem the first term vanishes apart from boundary terms (since $\alpha_i \equiv 0$ over the boundary $\partial\Omega$ of Ω unless node i is on the boundary of the domain). There remains the term

$$\int_\Omega D\underline{\nabla}U(\alpha_i\underline{\nabla}U_x + U_x\underline{\nabla}\alpha_i)d\Omega$$

$$= \int_\Omega DU_x\underline{\nabla}U\cdot\underline{\nabla}\alpha_i d\Omega + \int_\Omega D\alpha_i \frac{\partial}{\partial x}[\tfrac{1}{2}(\underline{\nabla}U)^2]d\Omega$$

$$= \int_\Omega DU_x\underline{\nabla}U\cdot\underline{\nabla}\alpha_i d\Omega + \int_\Omega \frac{\partial}{\partial x}(D\alpha_i\cdot\tfrac{1}{2}(\underline{\nabla}U)^2)d\Omega - \int_\Omega (D\alpha_i)x\cdot\tfrac{1}{2}(\underline{\nabla}U)^2 d\Omega$$

Using integration by parts the second term vanishes apart from boundary terms, giving

$$\int_\Omega [DU_x\underline{\nabla}U\cdot\underline{\nabla}\alpha_i - \tfrac{1}{2}(D\alpha_i)_x(\underline{\nabla}U)^2]d\Omega$$

A similar analysis holds for the inner product $\langle\underline{\nabla}\cdot(D\underline{\nabla}U),\gamma_i\rangle$ and we have finally that

$$\langle\underline{\nabla}\cdot(D\underline{\nabla}U), \begin{matrix}\beta_i\\ \gamma_i\end{matrix}\rangle = \int_\Omega [D\underline{\nabla}U(\underline{\nabla}U\cdot\underline{\nabla}\alpha_i) - \tfrac{1}{2}\underline{\nabla}(D\alpha_i)(\underline{\nabla}U)^2]d\Omega \qquad (5.4)$$

+ boundary terms if node i is on the boundary.

Note that since Green's Theorem has been used we have implicitly assumed an arbitrary smoothing since $\underline{\nabla}U$ must be continuous across the element edges within the patch Ω in order to use Green's Theorem.

We now return to the inner products (5.2) and rewrite them in the form

$$\langle V\underline{\nabla}^2 V + \frac{1}{n}(\underline{\nabla}V)^2, \begin{matrix}\alpha_i\\ \beta_i\\ \gamma_i\end{matrix}\rangle = \langle\underline{\nabla}\cdot(V\underline{\nabla}V) + (\frac{1}{n}-1)(\underline{\nabla}V)^2, \begin{matrix}\alpha_i\\ \beta_i\\ \gamma_i\end{matrix}\rangle \qquad (5.5)$$

$$= \langle\underline{\nabla}\cdot(V\underline{\nabla}V), \begin{matrix}\alpha_i\\ \beta_i\\ \gamma_i\end{matrix}\rangle + (\frac{1}{n}-1)\langle(\underline{\nabla}V)^2, \begin{matrix}\alpha_i\\ \beta_i\\ \gamma_i\end{matrix}\rangle$$

The second term in (5.5) presents no difficulty since $(\underline{\nabla}V)^2$ is simply piecewise constant over each element. In the problem under consideration we evaluate the first term using (5.4) with $D(V) = V$. In this case we require to integrate piecewise linear functions over each element which may be done exactly using a three point quadrature.

We shall now proceed to derive equations for the velocity of the moving interface in one and two dimensions.

5.2 Interface Velocity in the One Dimensional Problem

Using the change of variable $v = u^n$ discussed in Section 3 we require to solve the hyperbolic equation (3.3) only at the moving interface $v = 0$. Thus in dimension we require to solve

$$v_t = \frac{1}{n} v_x^2 \quad . \tag{5.6}$$

We seek a piecewise linear solution of (5.6) in the form

$$V = \sum V_j \alpha_j(x, \underline{s}(t)) \quad .$$

Differentiating with respect to time yields

$$V_t = \sum \dot{V}_j \alpha_j + \dot{s}_j \beta_j \quad . \tag{5.7}$$

We may also represent the piecewise constant term on the RHS of (5.6) in terms of the basis functions α_j, β_j since

$$V_x^2 = \sum - m_j m_{j+1} \alpha_j - (m_j + m_{j+1}) \beta_j \tag{5.8}$$

where

$$m_j = \frac{V_j - V_{j-1}}{s_j - s_{j-1}} = \frac{\Delta V_j}{\Delta s_j} \quad , \text{ say.}$$

Substituting (5.7) and (5.8) into (5.6) gives

$$\sum \dot{V}_j \alpha_j + \dot{s}_j \beta_j = \frac{1}{n} \sum - m_j m_{j+1} \alpha_j - (m_j + m_{j+1}) \beta_j$$

and hence

$$\sum (\dot{V}_j + \frac{1}{n} m_j m_{j+1}) \alpha_j + (\dot{s}_j + \frac{1}{n} (m_j + m_{j+1})) \beta_j = 0 \quad . \tag{5.9}$$

Since our approximations to both sides of equation (5.6) lie in the space $S_{\alpha\beta}$ spanned by the basis functions α_j, β_j and since the functions β_j are discontinuous at the nodes we have an exact solution of (5.6) in semi-discrete form given by

$$\dot{V}_j = -\frac{1}{n} m_j m_{j+1}$$

$$\dot{s}_j = -\frac{1}{n} m_j m_{j+1} \qquad (5.10)$$

We now consider the node s_N at the moving interface $V = 0$, as shown in Fig. 5.1

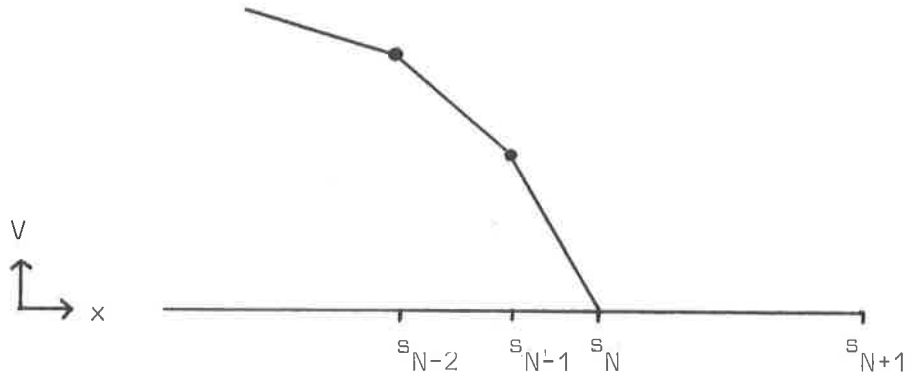

FIG. 5.1

If we set $j = N$ in (5.10) with $m_{n+1} = 0$ then we have

$$\dot{V}_N = 0$$

$$\dot{s}_N = -\frac{1}{n} m_N \qquad (5.11)$$

Equation (5.11) gives the velocity of the interface exactly in semi-discrete form and we may replace the MFE equations resulting from minimisation of the residual with respect to $\dot{V}_N$ and $\dot{s}_N$ by equations (5.11).

It is also found to be desirable to impose the interface velocity on the node s_{N-1} immediately before the interface. This is because we use a quadratic recovery procedure to interpret inner products of the form $\langle VV_{xx}, \beta_i \rangle$ on the RHS of the MFE equations, and fitting a quadratic to the gradient V_x over the element (s_{N-1}, s_N) may give a spurious result due to the large discontinuity in V_x at the interface. The remaining inner product $\langle VV_{xx}, \alpha_{N-1} \rangle$ for node s_{N-1} may be evaluated using integration by parts.

5.3 Interface Velocity in Two Dimensions

In two dimensions at the moving interface we require to solve

$$v_t = \frac{1}{n}(\underline{\nabla} v)^2 \quad . \tag{5.12}$$

We seek a piecewise linear solution of (5.12) in the form

$$V = \sum V_j \alpha_j(x,y,\underline{r}(t)) \quad .$$

We first consider two elements either side of the moving interface as shown in Fig. 5.2 below.

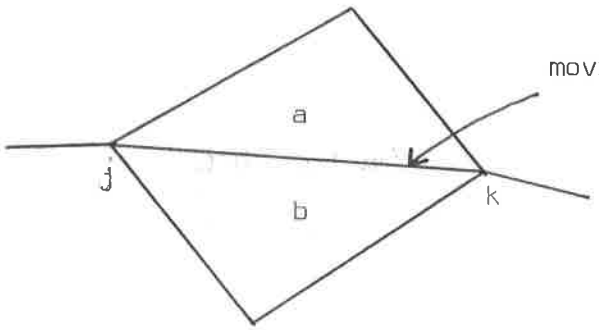

FIG. 5.2

We shall derive expressions for both sides of (5.12) on elements a and b and in particular we shall look at the form of these expressions as we approach the interface jk from within each element.

In element a we have

$$V_t = \sum_{i\in(j,k,\ell)} \dot{V}_i \alpha_i + \dot{X}_i \beta_i + \dot{Y}_i \gamma_i$$

and in particular along jk

$$V_t = \sum_{i\in(j,k)} \dot{V}_i \alpha_i + \dot{X}_i \beta_i + \dot{Y}_i \gamma_i \quad .$$

In element a let $V_x = V_x^a$, $V_y = V_y^a$ then along jk

$$V_t = \sum_{i\in(j,k)} (\dot{V}_i - V_x^a \dot{X}_i - V_y^a \dot{Y}_i)\alpha_i$$

$$= (\dot{V}_j - V_x^a \dot{X}_j - V_y^a \dot{Y}_j)\phi_{aj} + (\dot{V}_k - V_x^a \dot{X}_k - V_y^a \dot{Y}_k)\phi_{ak} \tag{5.13}$$

where ϕ_{aj}, ϕ_{ak} are the element basis functions shown in Fig. 5.3

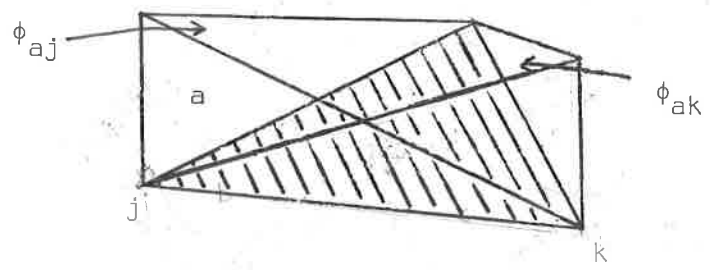

FIG. 5.3

In element a along the line jk we may express the piecewise constant RHS of (5.12) in the form

$$\frac{1}{n}(\underline{\nabla}V)^2 = \frac{1}{n}(V_x^{a^2} + V_y^{a^2})(\phi_{aj} + \phi_{ak}) \tag{5.14}$$

Substituting (5.13) and (5.14) into (5.12) yields

$$(\dot{V}_j - V_x^a\dot{X}_j - V_y^a\dot{Y}_j)\phi_{aj} + (\dot{V}_k - V_x^a\dot{X}_k - V_y^a\dot{Y}_k)\phi_{ak}$$

$$= \frac{1}{n}(V_x^{a^2} + V_y^{a^2})(\phi_{aj} + \phi_{ak}) \quad . \tag{5.15}$$

Since the RHS of (5.15) is a constant the solution of (5.15) is

$$\dot{V}_j - V_x^a\dot{X}_j - V_y^a\dot{Y}_j = \dot{V}_k - V_x^a\dot{X}_k - V_y^a\dot{Y}_k = \frac{1}{n}(V_x^{a^2} + V_y^{a^2}) \tag{5.16}$$

Moreover, since $V \equiv 0$ on element a $V_x^a = 0$ and $V_y^a = 0$, and hence from (5.16)

$$\dot{V}_j = \dot{V}_k = 0 \quad . \tag{5.17}$$

A similar analysis in element b (see Fig. 5.2), together with equations (5.17) yields

$$-V_x^b\dot{X}_j - V_x^b\dot{Y}_j = -V_x^b\dot{X}_k - V_x^b\dot{Y}_k = \frac{1}{n}(V_x^{b^2} + V_y^{b^2}) \tag{5.18}$$

Applying the same analysis on element c shown in Fig. (5.4), gives

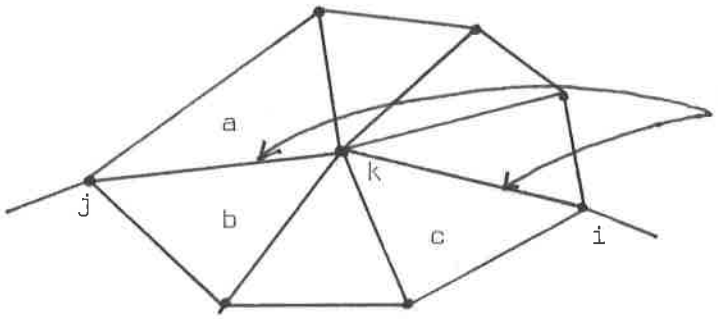

FIG. 5.4

$$-V_x^c \dot{X}_k - V_x^c \dot{Y}_k = - V_x^c \dot{X}_i - V_x^c \dot{Y}_i = \frac{1}{n} (V_x^{c^2} + V_y^{c^2}) \quad . \tag{5.19}$$

We now have two equations for the interface velocities $\dot{X}_k$, $\dot{Y}_k$ given by (5.18) and (5.19), which we may proceed to solve. Let $M_x = V_x^b$, $M_y = V_y^b$, $\bar{M}_x = V_x^c$, $\bar{M}_y = V_y^c$. Then

$$- M_x \dot{X}_k - M_y \dot{Y}_k = \frac{1}{n} (M_x^2 + M_y^2)$$

and (5.20)

$$- \bar{M}_x \dot{X}_k - \bar{M}_y \dot{Y}_k = \frac{1}{n} (\bar{M}_x^2 + \bar{M}_y^2) \quad .$$

Solving (5.20), for $\dot{X}_k$, $\dot{Y}_k$ yields

$$\dot{X}_k = \frac{1}{n} \left(\frac{1}{M_x \bar{M}_y - M_y \bar{M}_x} \right) (-\bar{M}_y (M_x^2 + M_y^2) + M_y (\bar{M}_x^2 + \bar{M}_y^2))$$

$$\dot{Y}_k = \frac{1}{n} \left(\frac{1}{M_x \bar{M}_y - M_y \bar{M}_x} \right) (\bar{M}_x (M_x^2 + M_y^2) - M_x (\bar{M}_x^2 + \bar{M}_y^2)) \tag{5.21}$$

We replace the MFE equations for the velocities of all nodes lying on the moving interface by the interface velocities given by (5.21), and those for the amplitudes by equations (5.17).

6. A MAXIMUM PRINCIPLE FOR THE GRADIENT OF THE MFE SOLUTION

In this section we derive a maximum principle for the gradient of the MFE solution of (2.1) in one dimension. As usual we seek an approximate solution of the transformed problem

$$v_t = vv_{xx} + \frac{1}{n} vx^2 \tag{6.1}$$

in the form

$$V = \sum V_j \alpha_j(x, \underline{s}(t)) \quad .$$

As mentioned in Section 5, in order to evaluate inner products of the form $\langle VV_{xx}, \begin{matrix}\alpha_i \\ \beta_i\end{matrix} \rangle$ which occur on the RHS of the MFE equations we recover from the gradient V_x of the MFE solution a quadratic w_x defined on element i by

$$\begin{aligned} w_x(s_{i-1}) &= \tfrac{1}{2}(M + M_L) & m &= \frac{\Delta V_i}{\Delta s_i} \\ w_x(\tfrac{1}{2}(s_i + s_{i-1})) &= M & m_L &= \frac{\Delta V_{i-1}}{\Delta s_{i-1}} \\ w_x(s_i) &= \tfrac{1}{2}(M + M_R) & m_R &= \frac{\Delta V_{i+1}}{\Delta s_{i+1}} \end{aligned} \tag{6.2}$$

Hence, instead of solving (6.1) we are in effect solving

$$v_t = vw_{xx} + \frac{1}{n} v_x^2 \quad . \tag{6.3}$$

It may be shown (see Baines [2]) that the gradient m_k of the MFE solution on element k satisfies the ordinary differential equation

$$\frac{dm_k}{dt} = \frac{6}{(\Delta s_k)^2} \int_{s_{k-1}}^{s_k} \psi_k (vw_{xx} + \frac{1}{n} v_x^2) dx \tag{6.4}$$

where ψ_k is as shown in Fig. 6.1 overleaf.

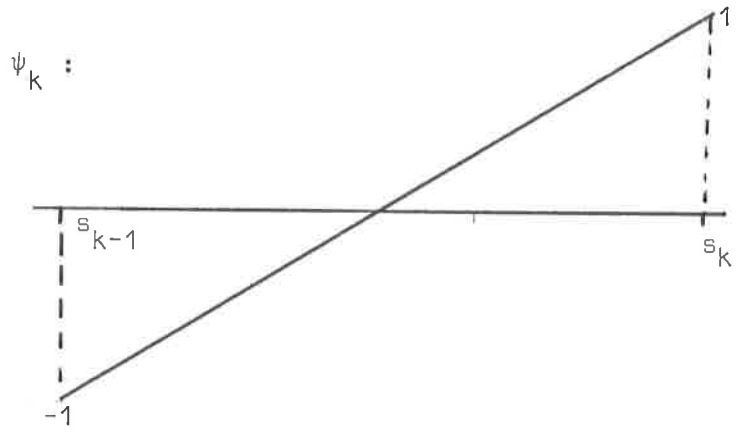

FIG. 6.1

Since V_x is constant over element k

$$\int_{s_{k-1}}^{s_k} \psi_k \left(\frac{1}{n} V_x^2\right) dx = 0$$

and hence (6.4) reduces to

$$\frac{dm_k}{dt} = \frac{6}{(\Delta s_k)^2} \int_{s_{k-1}}^{s_k} \psi_k \, V w_{xx} \, dx \qquad (6.5)$$

The functions ψ_k, V and w_{xx} inside the integral are all linear over element k and hence we need only to integrate a cubic over the element. After some lengthy calculation we have the result

$$\int_{s_{k-1}}^{s_k} \psi_k V w_{xx} \, dx = \frac{1}{12} \left\{ (V_k + 3V_{k-1}) m_L - 4(V_k + V_{k-1}) m + (V_{k-1} + 3V_k) m_R \right\}$$

where $$m = m_k = \frac{\Delta V_k}{\Delta s_k} \quad , \quad m_L = \frac{\Delta V_{k-1}}{\Delta s_{k-1}} \quad , \quad m_R = \frac{\Delta V_{k+1}}{\Delta s_{k+1}}$$

and hence in (6.5)

$$\frac{dm_k}{dt} = \frac{1}{2(\Delta s_k)^2} \left\{ (V_k + 3V_{k-1}) m_L - 4(V_k + V_{k-1}) m + (V_{k-1} + 3V_k) m_R \right\} \quad . \qquad (6.6)$$

Explicit Euler time stepping in (6.6) at time level t^n yields

$$m^{n+1} = m^n + \frac{\Delta t}{2(\Delta s_k)^2}\left\{(V_k+3V_{k-1})m_L^n - 4(V_k+V_{k-1})m^n + (V_{k-1}+3V_k)m_R^n\right\}$$

$$= m^n\left(1 - \frac{2\Delta t}{(\Delta s_k)^2}(V_k+V_{k-1})\right) + \frac{\Delta t}{2(\Delta s_k)^2}\left((V_k+3V_{k-1})m_L^n + (V_{k-1}+3V_k)m_R^n\right) \tag{6.7}$$

From the nature of the problem we have

$$V_k \geq 0 \qquad \forall_k \ .$$

and $$m_L,\ m_R,\ m \leq 0$$

Hence if we define $M = \min(m^n, m_L^n, m_R^n)$ and if

$$1 - \frac{2\Delta t}{(\Delta s_k)^2}(V_k + V_{k-1}) > 0 \quad \text{i.e.} \quad \Delta t < \frac{(\Delta S_k)^2}{2(V_k+V_{k-1})} \tag{6.8}$$

then subject to condition (6.8) we have from (6.7)

$$m^{n+1} \geq M\left(1 - \frac{2\Delta t}{(\Delta s_k)^2}(V_k+V_{k-1})\right) + \frac{\Delta t}{2(\Delta s_k)^2}((V_k+3V_{k-1})M + (V_{k-1}+3V_k)M)$$

and hence $$m^{n+1} \geq M$$

and the maximum principle subject to the time step restriction (6.8) is established. Using a similar analysis we may also establish a minimum principle.

7. TIME STEPPING

As mentioned in Section 4 we use explicit time stepping which, amongst other things, exploits the structure of the MFE matrix. We recall that the MFE matrix may be inverted explicitly in one dimension, and very efficiently using preconditioned Conjugate Gradients in higher dimensions. In the original formulation of the MFE method (see Miller & Miller [7]) penalty terms were added to prevent element folding since the MFE matrix becomes singular when the area of an element is zero (see Wathen & Baines [15]). Consequently the MFE equations became very stiff as the area of an element approached zero. As an alternative to the addition of penalty functions in the formulation of the method we have used a time stepping algorithm which restricts the size of the explicit time step to be such that element folding will not occur. The following argument applies to any number of dimensions.

7.1 Time Stepping Algorithm

If we solve the MFE equations using explicit Euler time stepping we have for the time step t^n to t^{n+1}

$$\underline{y}^{n+1} = \underline{y}^n + \Delta t^n A^{-1}(\underline{y}^n)\underline{g}(\underline{y}^n) \quad .$$

At the new time level t^{n+1} we check the sign of the Jacobian of each element to see whether element folding has occurred in moving from time level t^n to t^{n+1}. If folding has occurred we may compute the exact time $t^{n+*} \in [t^n, t^{n+1}]$ at which the area of an element is first equal to zero. In one dimension this involves solving a linear equation and in two dimensions a quadratic equation with coefficients depending on $\underline{y}^n$ and $\dot{\underline{y}}^n$. Having solved for t^{n+*} we now take a backward time step Δt_b of size

$$\Delta t_b = \Delta t^n - \theta(t^{n+*} - t^n) \qquad \theta \in (0,1) \quad .$$

We have used the value $\theta = \frac{1}{2}$ which advances the solution from time level t^n forward in time by an amount which is equal to half the time in which element folding would first occur, hence the new time level t^{n+1} is given by

$$t^{n+1} = t^n + \tfrac{1}{2}(t^{n+*} - t^n) \quad .$$

In practice it is found that the time step is never reduced successively more than two or three times, and reduction often occurs only in the initial stages of the computation, as the nodes readjust from their initial positions.

Adaptive Time Stepping

We have used a simple adaptive time stepping procedure in which the size of the time step is chosen such that the relative change in size of the diffusion coefficient v is restricted by some given constant value.

We select an appropriate constant θ and at time level t^n the step size Δt^n is given by

$$\Delta t^n = \frac{1}{\max_i \left\{ \frac{|\dot{V}_i^n|}{|V_i^n|} \right\}} \times \theta \qquad (\forall V_i > 0) \quad .$$

Typically we take $\theta = O(10^{-2})$ which restricts the relative change in the size of the diffusion coefficient over one time step to about 1%.

In the numerical results we give the average time step up to the output time and the size of the time step taken immediately prior to output.

8. NUMERICAL RESULTS

In this section we compare numerical results using MFE with the similarity solution derived in Section 2. For this problem the initial data is a fixed quantity of thermal energy initially at the origin, but in our computations we have started the process at time $t = 10^{-3}$ using the similarity solution to give the initial nodal values.

8.1 One Dimensional Results

In one dimension we have generally placed the nodes initially using an equidistributing principle (see Herbst [4], Johnson [5]), that is, given some initial data

$$u(x,t_0) = f(x)$$

we place the nodes such that the quantity

$$\int_{s_{i-1}}^{s_i} |f''(x)|^{\frac{1}{2}}\, dx \qquad (8.1)$$

is equidistributed over each element.

The similarity solution for the transformed variable v is given by

$$v(x,t) = \begin{cases} \dfrac{C^n}{R^n}\left[1 - \left(\dfrac{x}{R}\right)^2\right] & 0 \le x \le R(t) \\ 0 & r > R(t) \end{cases}$$

with C constant, which is a quadratic for all values of n.

Since $v(x,t)$ is quadratic, equidistributing the quantity (8.1) with $f(x) = v(x,t_0)$ is equivalent to equally spacing the nodes along the x axis for $x \in (0,R(t_0))$.

We have solved (2.1) for $x > 0$ and at the boundary $x = 0$ we have imposed the symmetry condition $V_x(0,t) = 0$.

Numerical results for the one dimensional examples are given in Figs. 8.1 to 8.3 overleaf.

MFE SOLUTION OF Ut= (UUx) x

Ux (0, t) =0

INITIAL DATA GIVEN BY SIMILARITY SOLN

NUMBER OF MOVING NODES= 10

OUTPUT AT EQUAL INTERVALS OF TIME

TSTART= 0.0010 TMAX= 0.0300

THETA= 0.005 INITIAL DT= 0.00000456

AVERAGE DT= 0.0000647

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MFE SOLUTION OF $U_t = (U^n U_x)_x$

$U_x(0,t) = 0$ $n = 3$

INITIAL DATA GIVEN BY SIMILARITY SOLN

USE SUBSTITUTION $V = U^n$ AND SOLVE

$V_t = VV_{xx} + V_x * V_x / n$ USING MFE

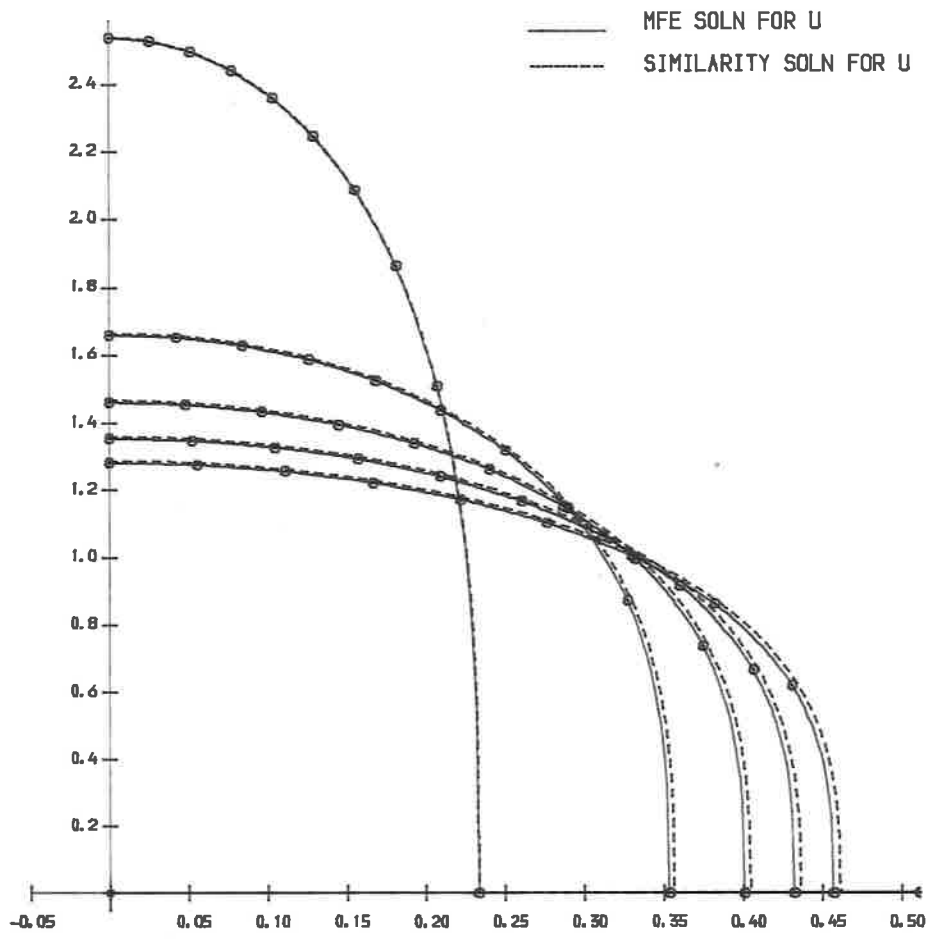
NUMBER OF MOVING NODES= 10

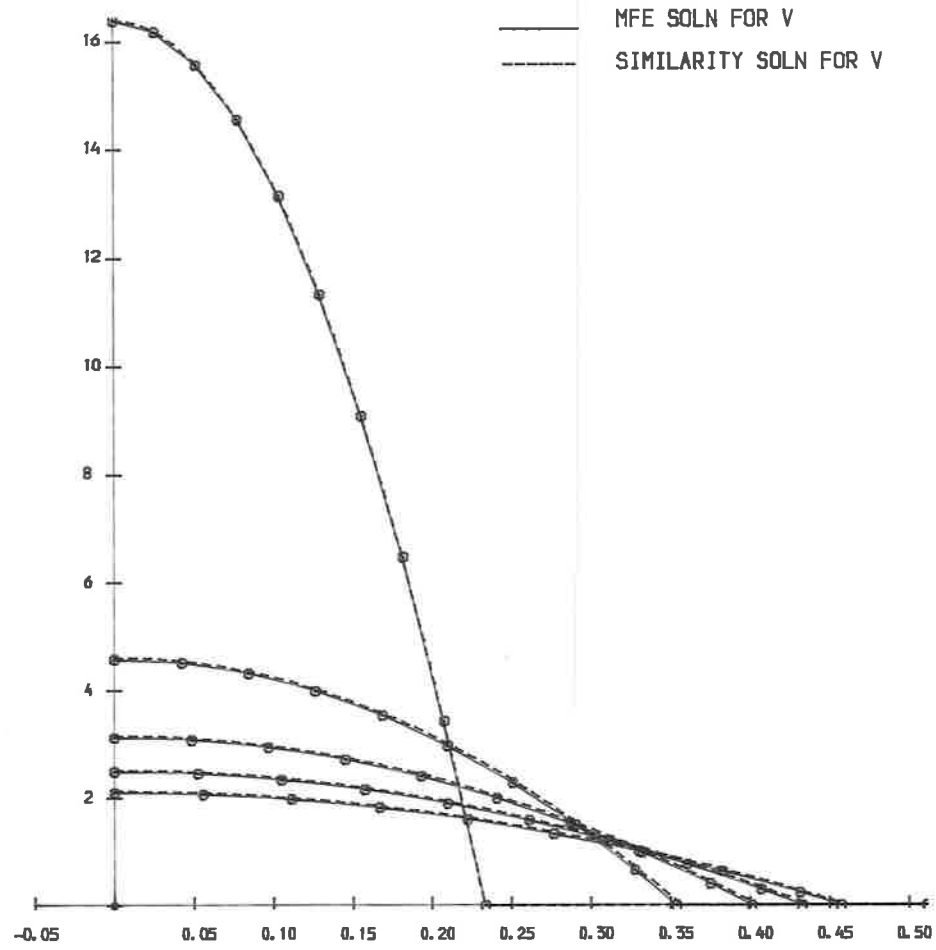
OUTPUT AT EQUAL INTERVALS OF TIME

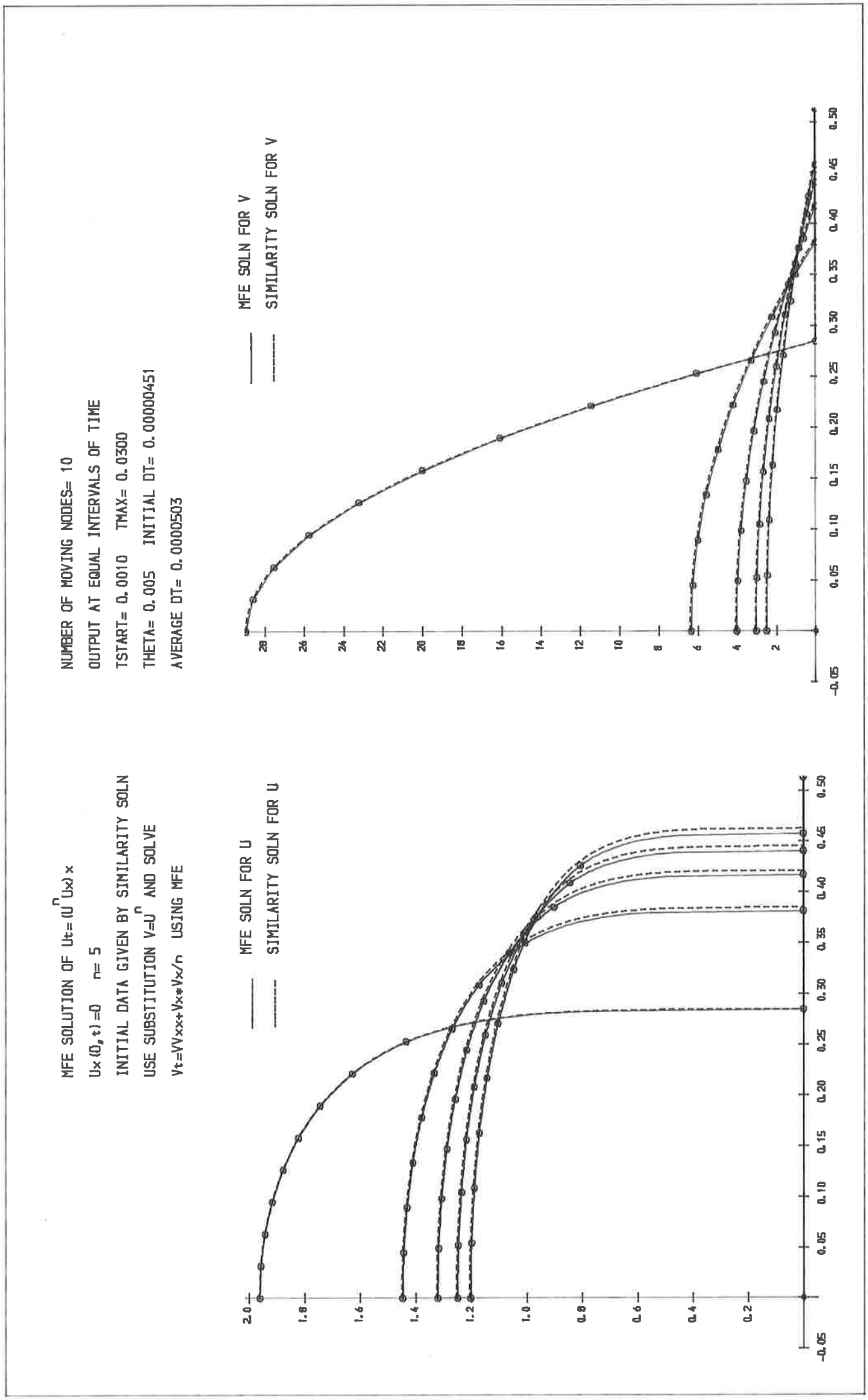
TSTART= 0.0010 TMAX= 0.0300

THETA= 0.005 INITIAL DT= 0.00000453

AVERAGE DT= 0.0000541







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8.2 Two Dimensional Results

In two dimensions the initial mesh is chosen such that the area of the triangles are approximately equal over the region in which the initial data is non-zero, as this corresponds to equidistributing a quantity analagous to (8.1) in two dimensions with $\underline{\nabla}^2 V(x,y,t_0)$ being constant.

Numerical results for two dimensional examples are given in the following figures.

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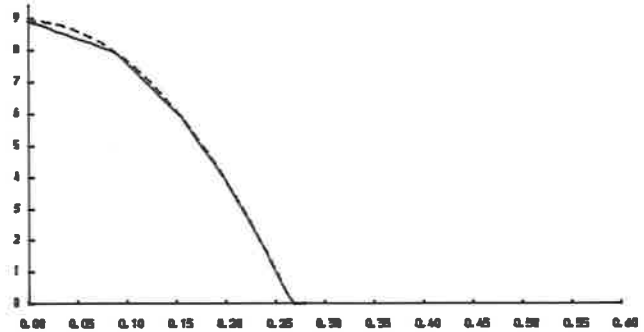
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 1 THETA= 0.005

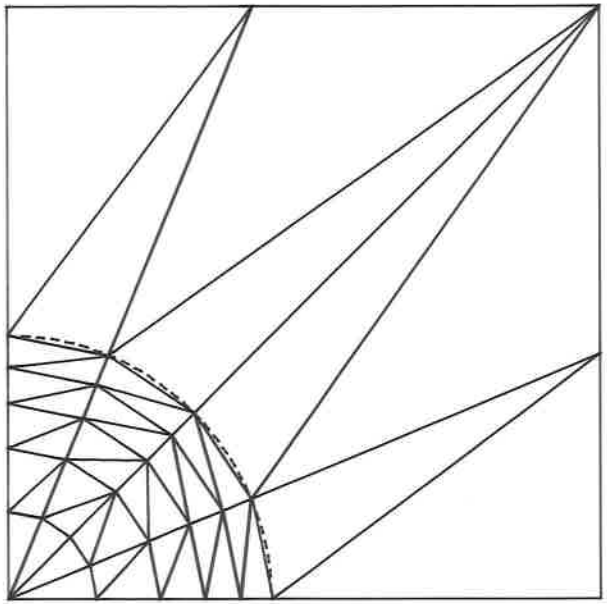
TSTART= 0.0010 INITIAL DT= 0.00000053

TIME STEP 0 OUTPUT TIME= 0.00100



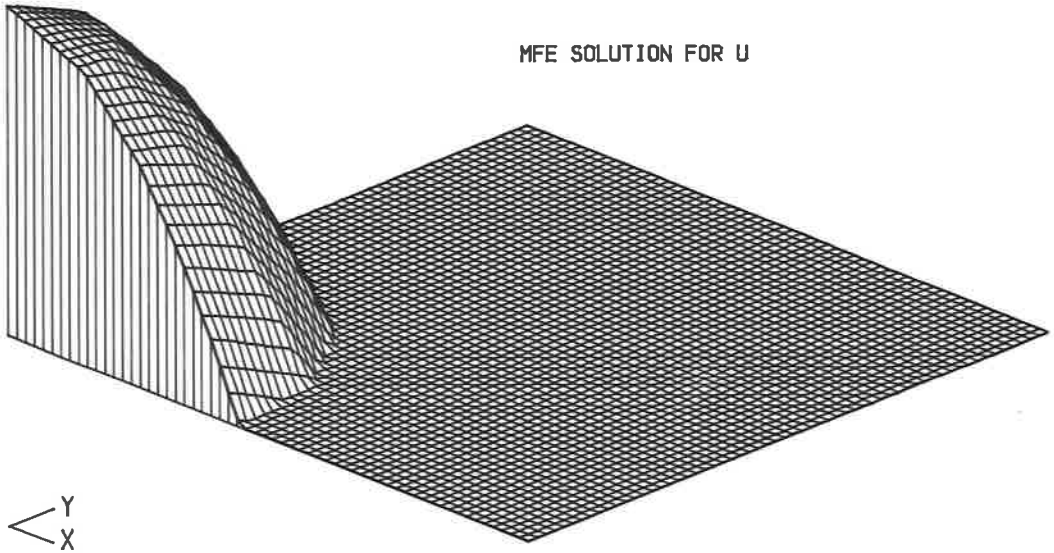
SOLUTIONS PLOTTED ALONG Y=X

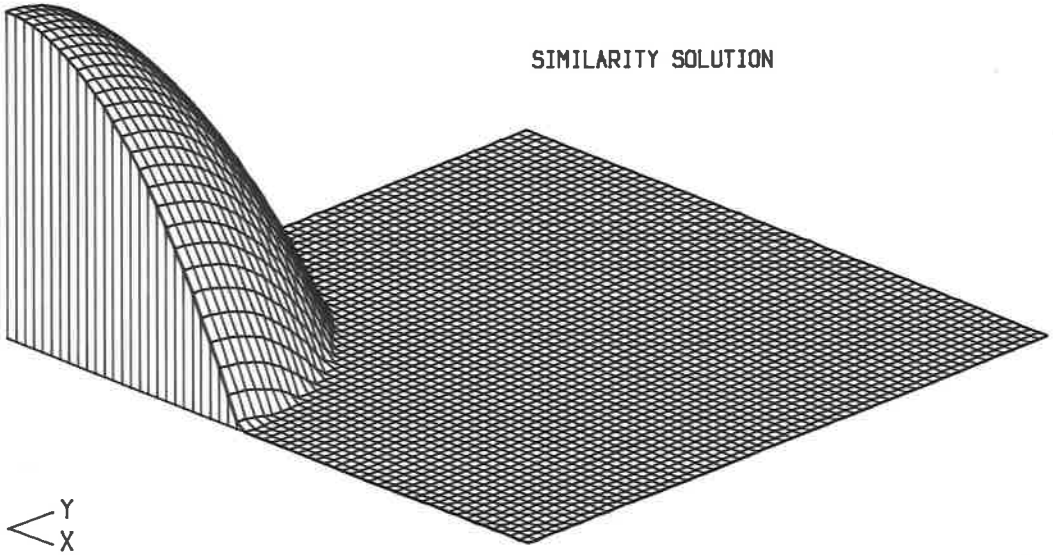
_____ MFE SOLUTION ----- SIMILARITY SOLUTION



_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION





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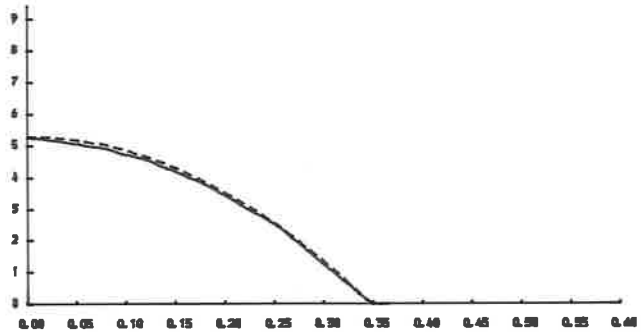
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 1 THETA= 0.005

TSTART= 0.0010 INITIAL DT= 0.00000053

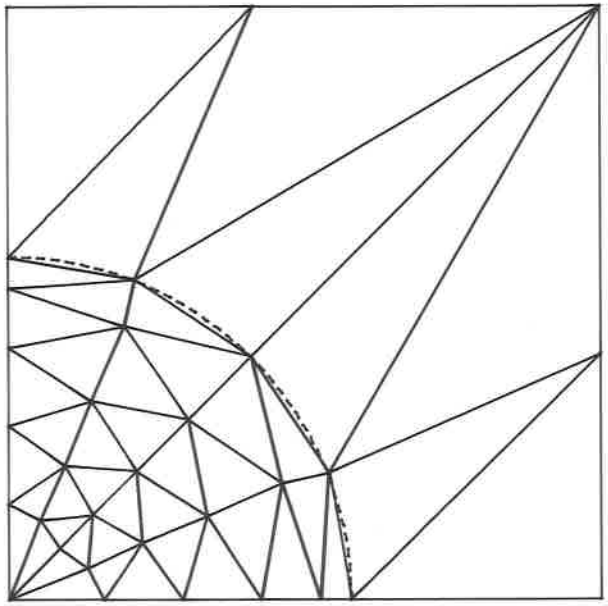
TIME STEP 300 OUTPUT TIME= 0.00287

PRESENT DT= 0.0000197 AVERAGE DT= 0.0000062



SOLUTIONS PLOTTED ALONG Y=X

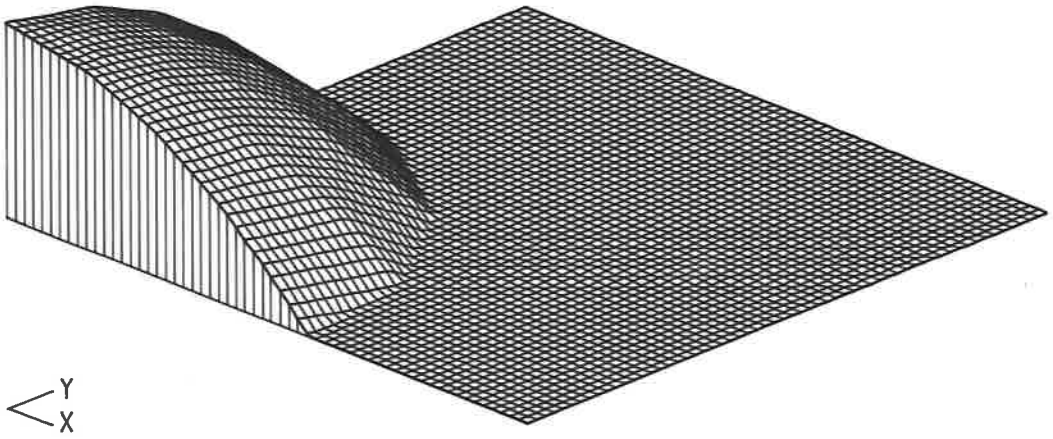
_____ MFE SOLUTION ------ SIMILARITY SOLUTION



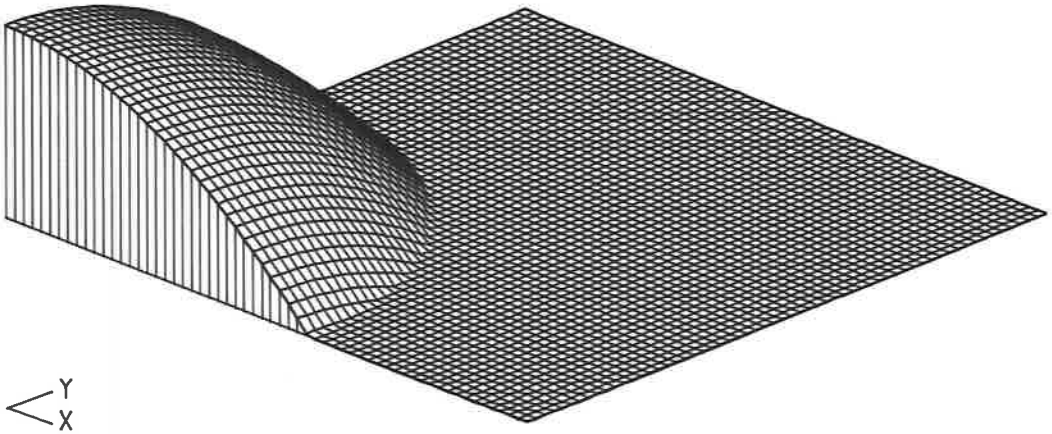
_____ MESH FOR MFE SOLUTION

------ POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION

MFE SOLUTION FOR U



SIMILARITY SOLUTION



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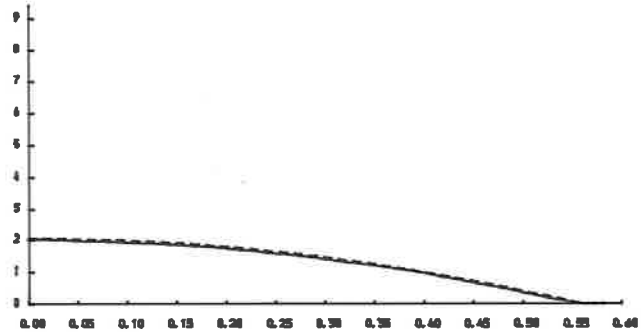
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 1 THETA= 0.005

TSTART= 0.0010 INITIAL DT= 0.00000053

TIME STEP 600 OUTPUT TIME= 0.01881

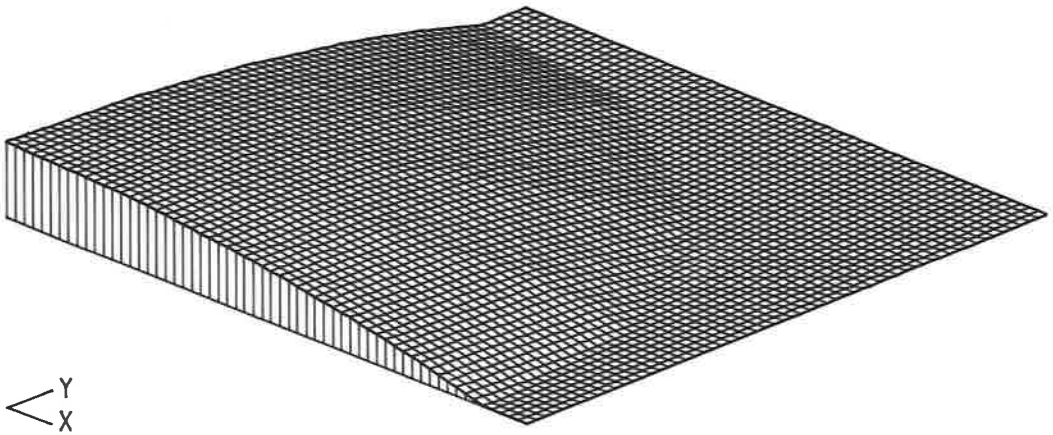
PRESENT DT= 0.0001192 AVERAGE DT= 0.0000297

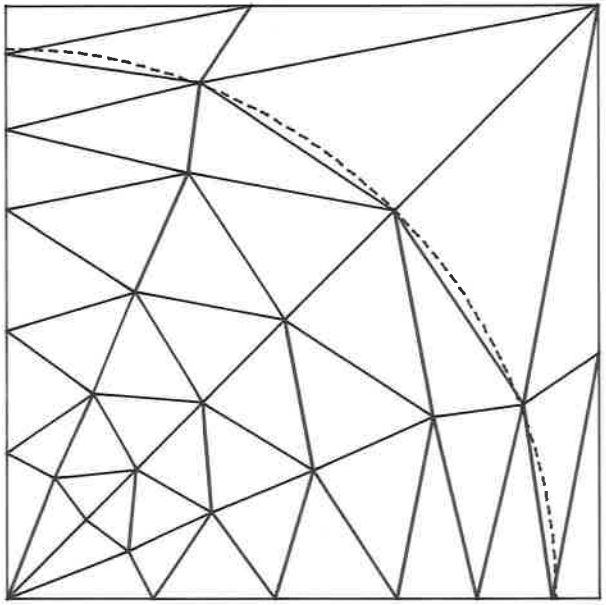


SOLUTIONS PLOTTED ALONG Y=X

_____ MFE SOLUTION ----- SIMILARITY SOLUTION

MFE SOLUTION FOR U

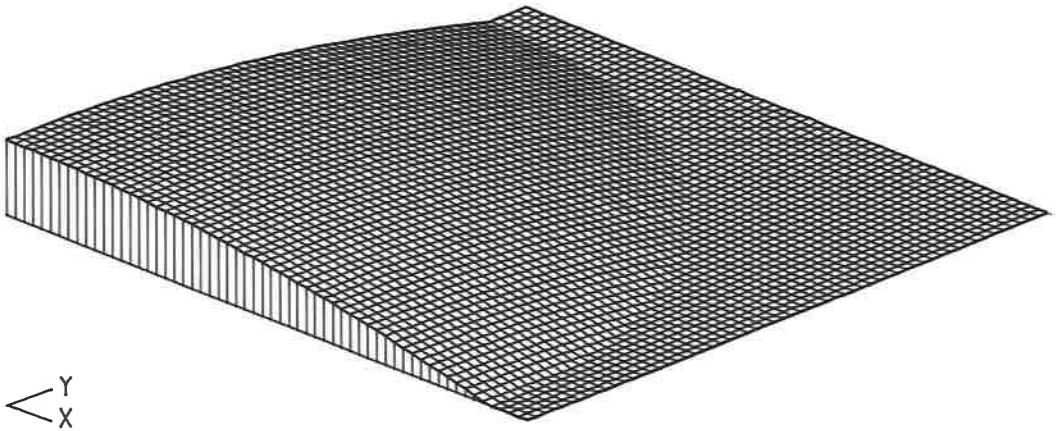




_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION

SIMILARITY SOLUTION



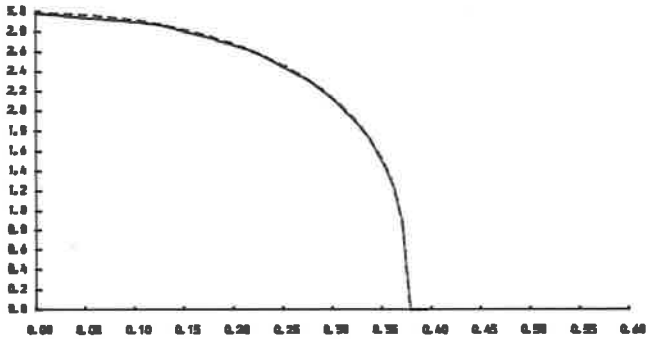
8450 AS
CS1917
CALCOMP EUROPE PRINTED IN U.K.

SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 3 THETA= 0.005

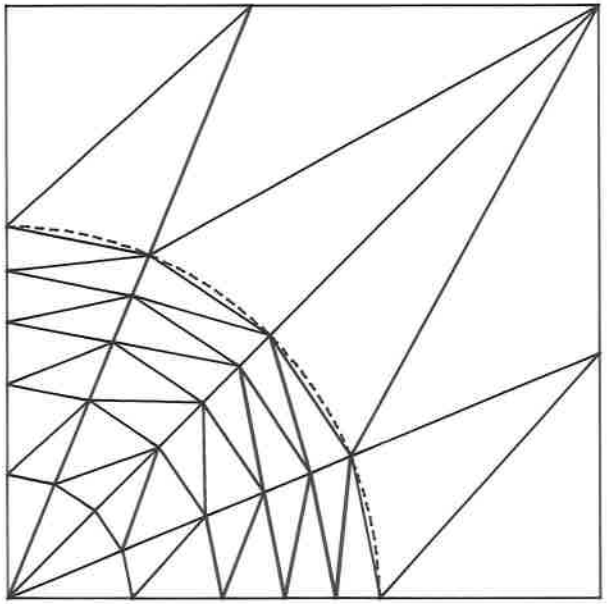
TSTART= 0.0010 INITIAL DT= 0.00000031

TIME STEP 0 OUTPUT TIME= 0.00100



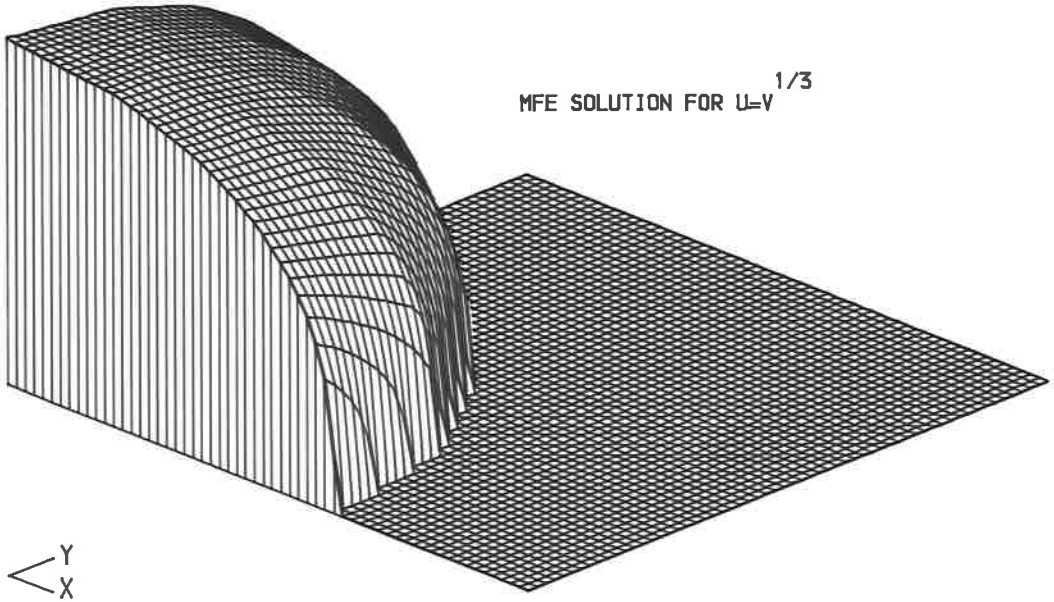
SOLUTIONS PLOTTED ALONG Y=X

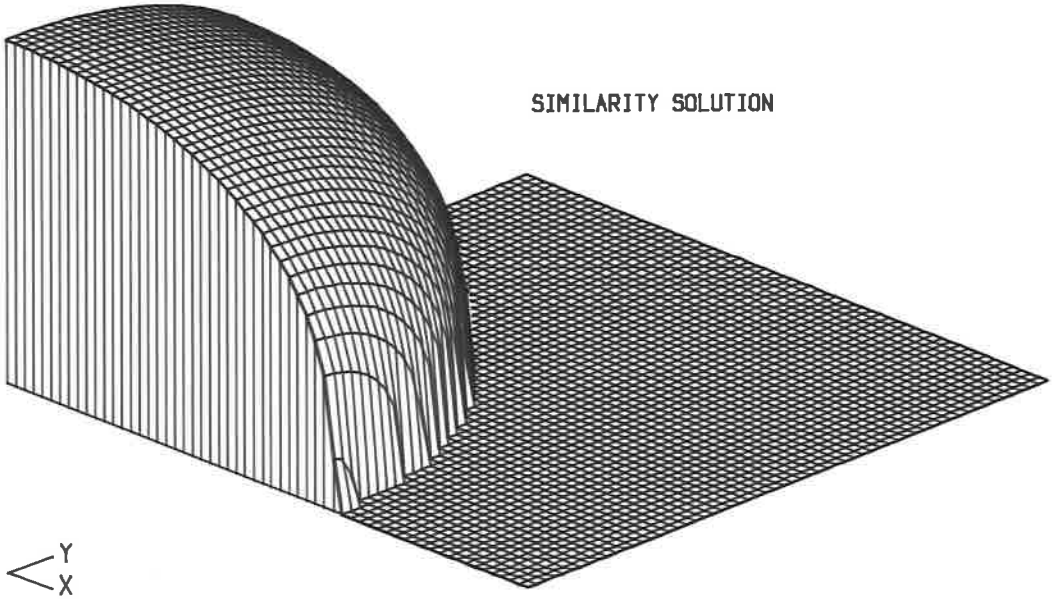
_____ MFE SOLUTION ----- SIMILARITY SOLUTION



_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION





CALCOMP EUROPE PRINTED IN U.K.
8450 AS

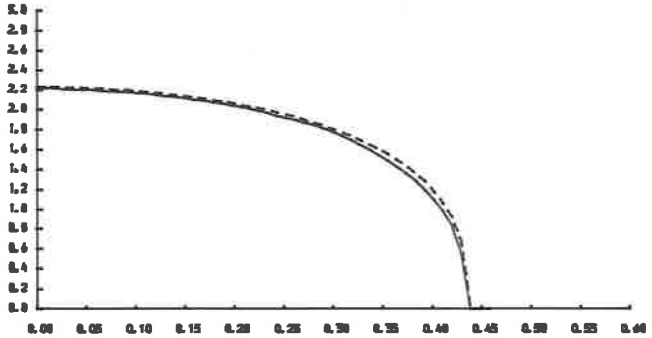
SOLUTION OF $U_t = \mathrm{div}(U^n \mathrm{grad}(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 3 THETA= 0.005

TSTART= 0.0010 INITIAL DT= 0.00000031

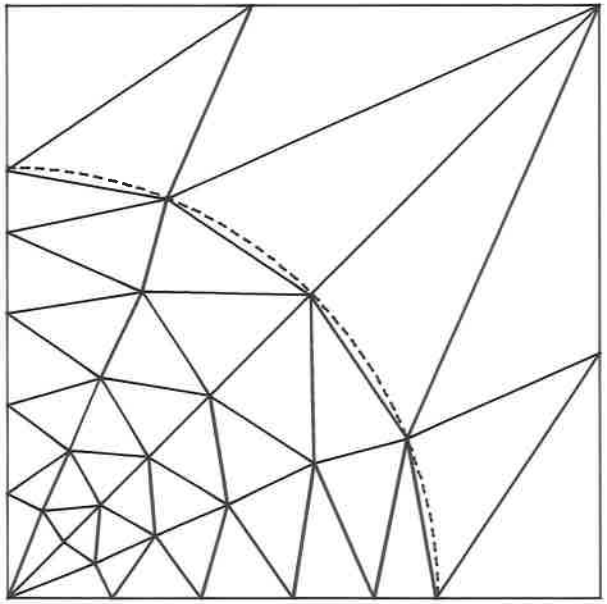
TIME STEP 400 OUTPUT TIME= 0.00323

PRESENT DT= 0.0000117 AVERAGE DT= 0.0000056



SOLUTIONS PLOTTED ALONG Y=X

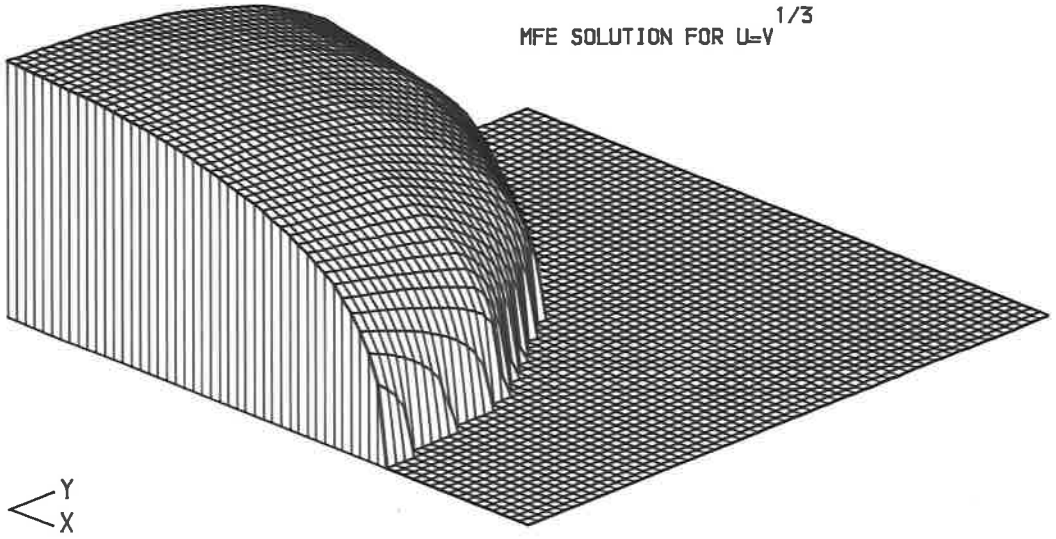
_____ MFE SOLUTION ----- SIMILARITY SOLUTION



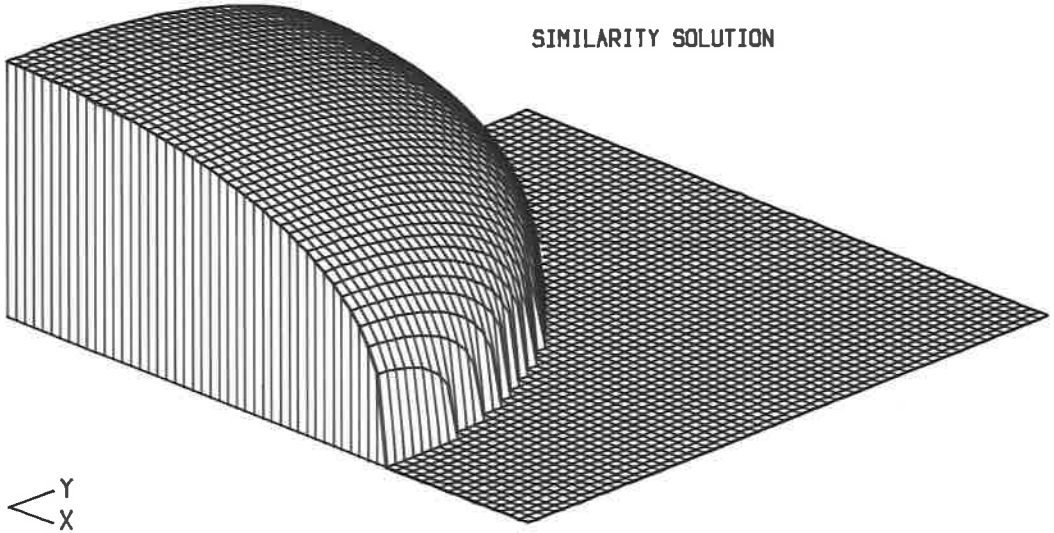
_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION

MFE SOLUTION FOR $U=V^{1/3}$



SIMILARITY SOLUTION



8450 AS

CALCOMP EUROPE PRINTED IN U.K.

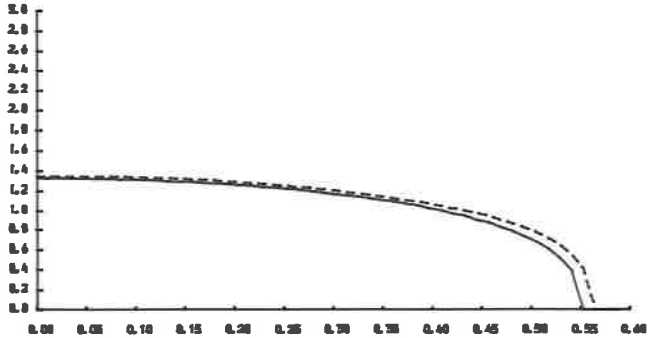
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 3 THETA= 0.005

TSTART= 0.0010 INITIAL DT= 0.00000031

TIME STEP 1000 OUTPUT TIME= 0.02427

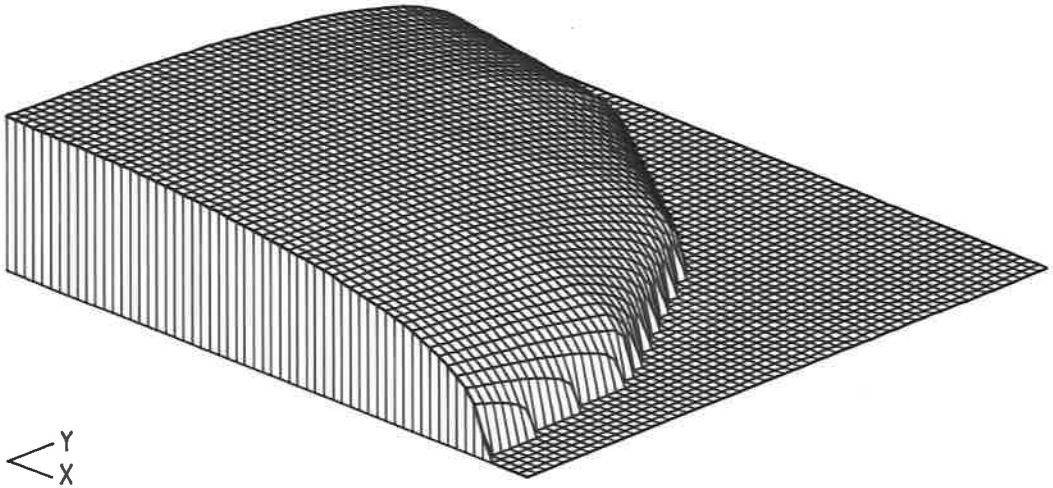
PRESENT DT= 0.0000860 AVERAGE DT= 0.0000233

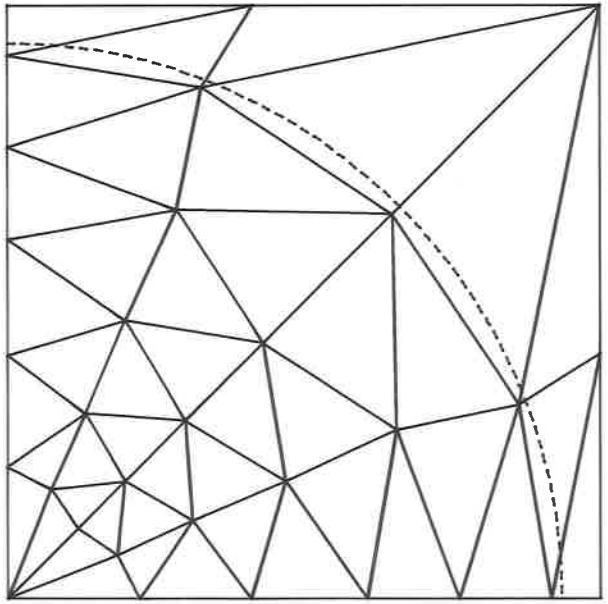


SOLUTIONS PLOTTED ALONG Y=X

_____ MFE SOLUTION ----- SIMILARITY SOLUTION

MFE SOLUTION FOR $U=V^{1/3}$

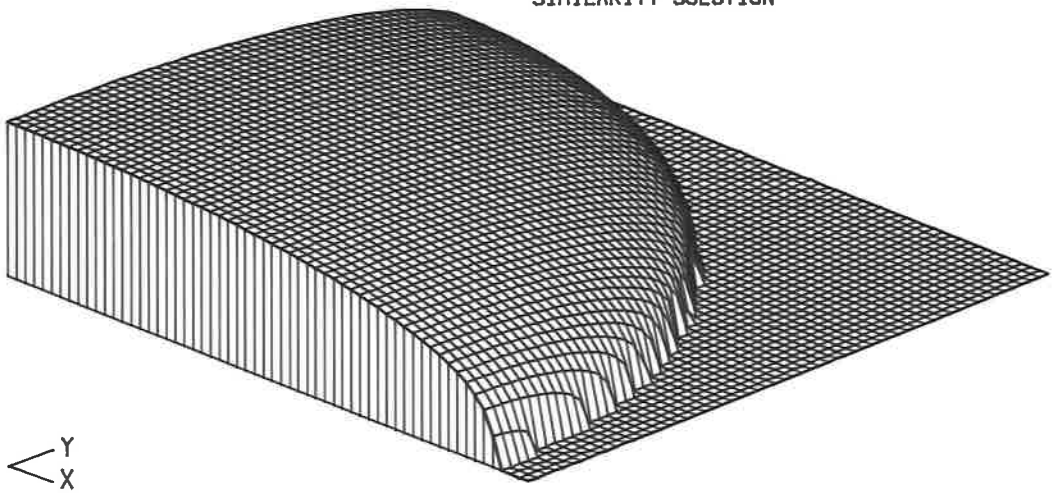




_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION

SIMILARITY SOLUTION



8450 AS

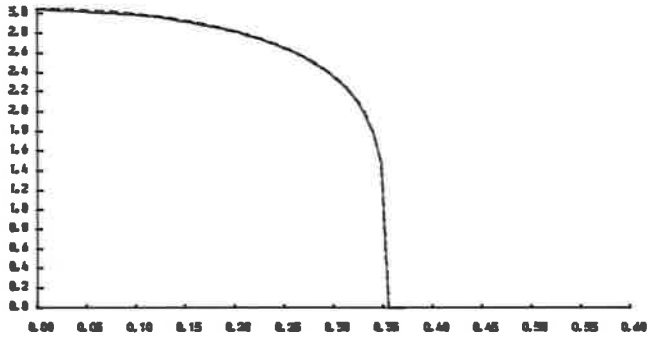
CALCOMP EUROPE PRINTED IN U.K.

SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 5 THETA= 0.005

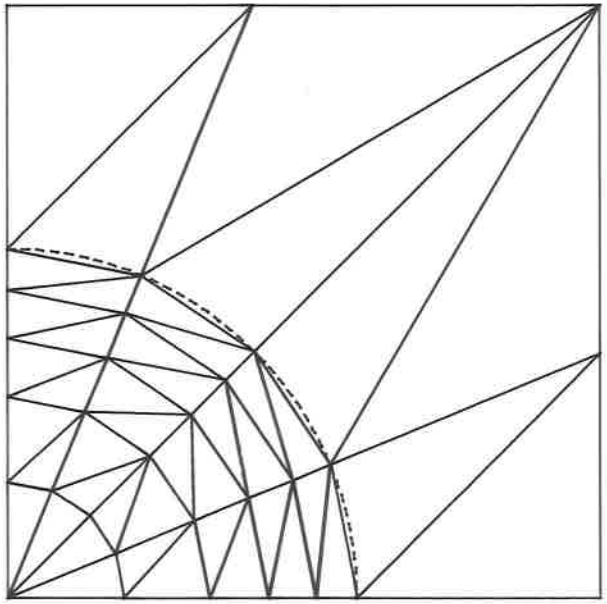
TSTART= 0.0001 INITIAL DT= 0.00000003

TIME STEP 0 OUTPUT TIME= 0.00010



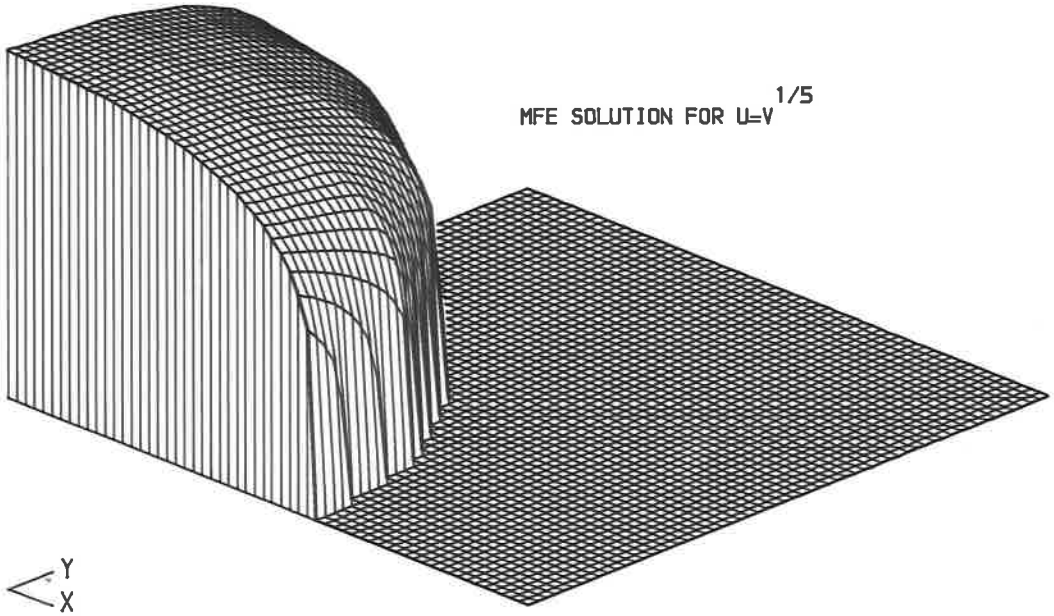
SOLUTIONS PLOTTED ALONG Y=X

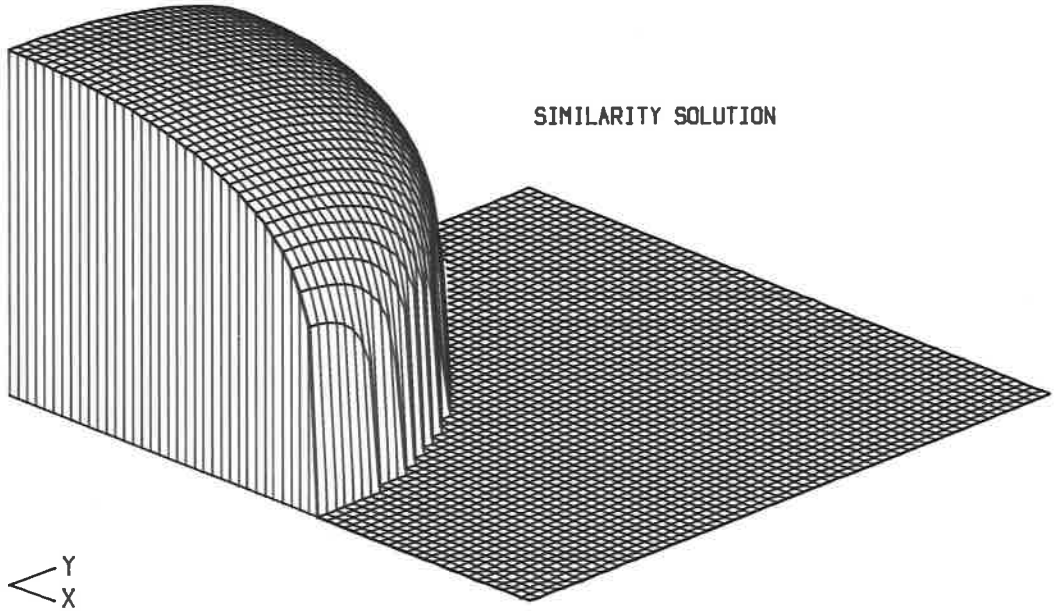
_____ MFE SOLUTION ----- SIMILARITY SOLUTION



_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION





8450 AS
CALCOMP EUROPE PRINTED IN U.K.

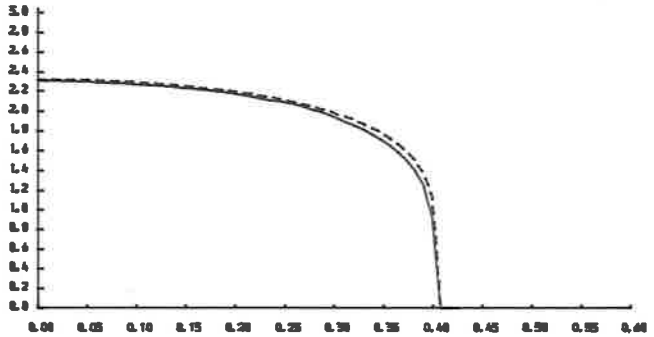
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 5 THETA= 0.005

TSTART= 0.0001 INITIAL DT= 0.00000003

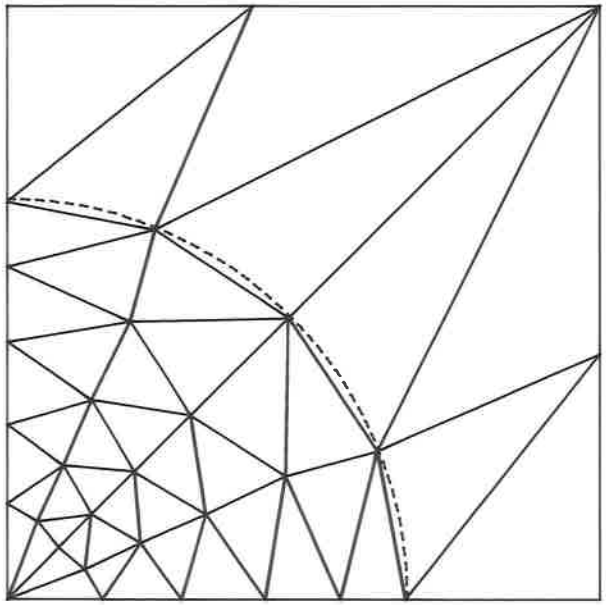
TIME STEP 600 OUTPUT TIME= 0.00051

PRESENT DT= 0.0000014 AVERAGE DT= 0.0000007



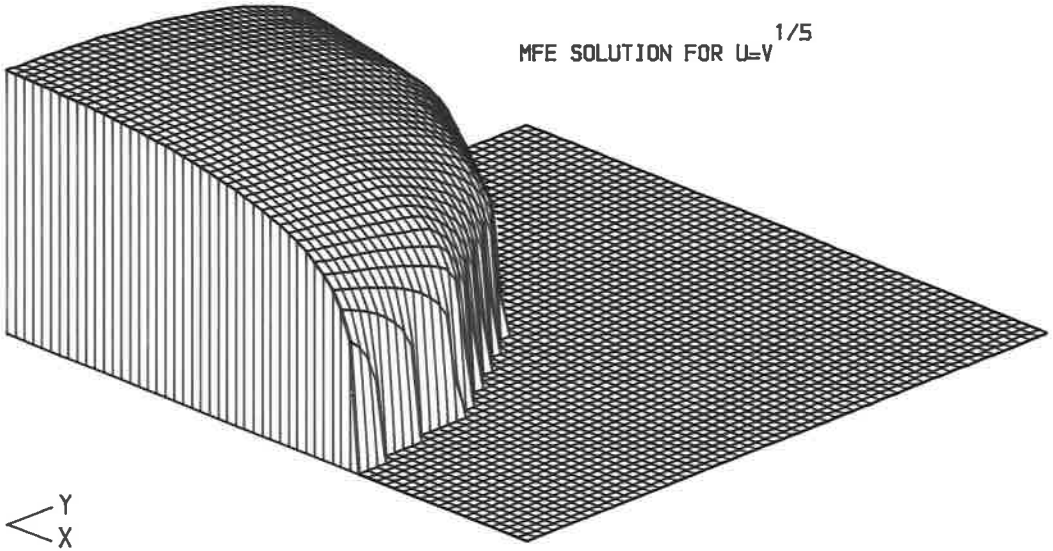
SOLUTIONS PLOTTED ALONG Y=X

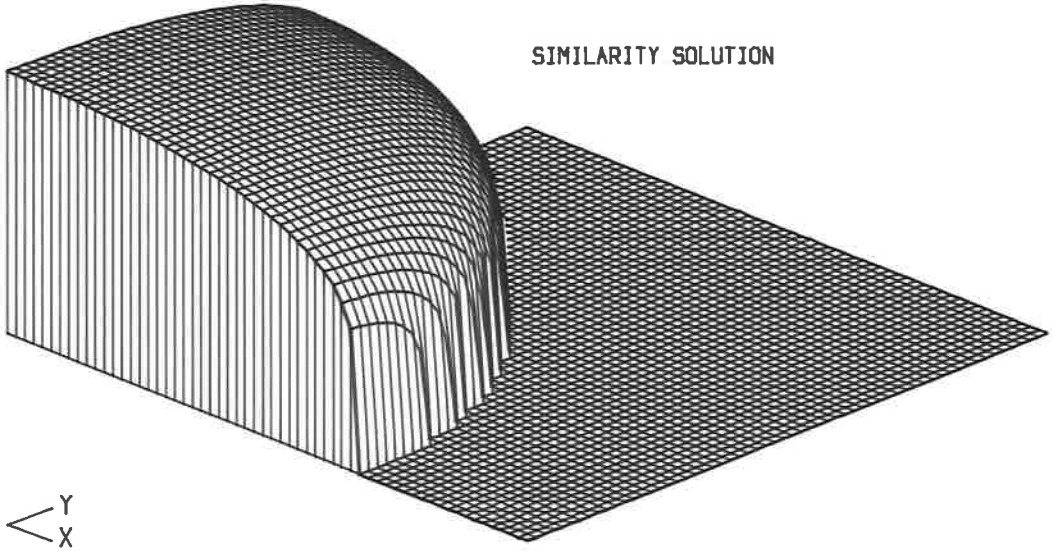
_____ MFE SOLUTION ----- SIMILARITY SOLUTION



_____ MESH FOR MFE SOLUTION

----- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION





8450 AS
CS1917
CALCOMP EUROPE PRINTED IN U.K.

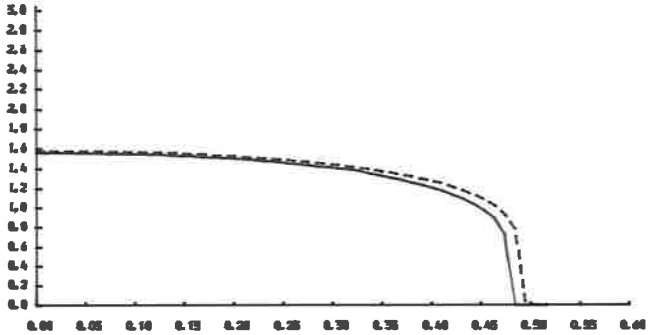
SOLUTION OF $U_t = div(U^n grad(U))$ SUBJECT TO CONSTANT TOTAL HEAT ENERGY

USING MFE WITH 31 NODES, 44 ELEMENTS n= 5 THETA= 0.005

TSTART= 0.0001 INITIAL DT= 0.00000003

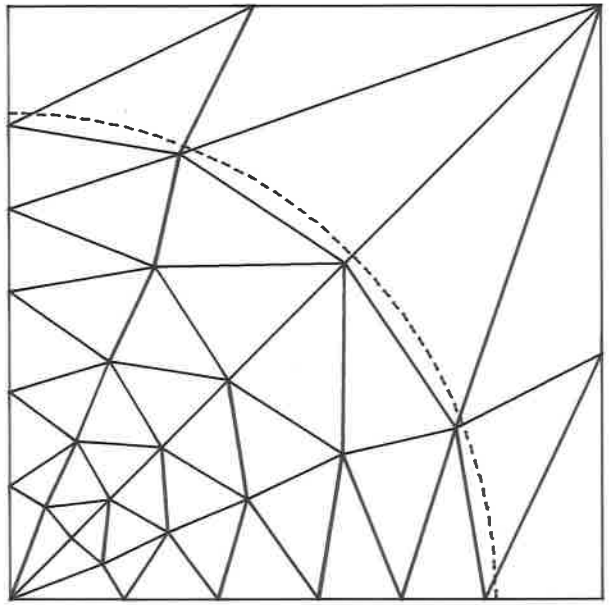
TIME STEP 1400 OUTPUT TIME= 0.00521

PRESENT DT= 0.0000141 AVERAGE DT= 0.0000036



SOLUTIONS PLOTTED ALONG Y=X

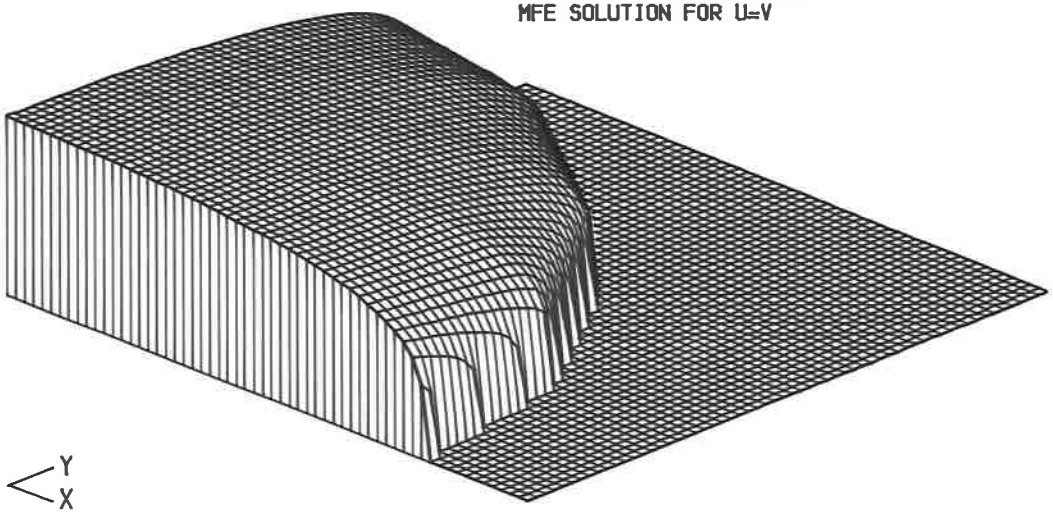
____ MFE SOLUTION ---- SIMILARITY SOLUTION



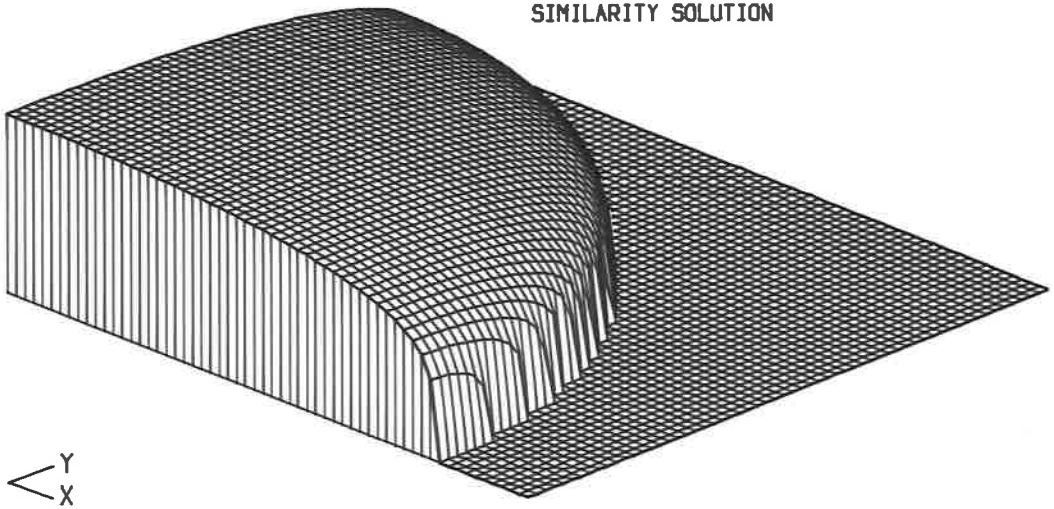
____ MESH FOR MFE SOLUTION

---- POSITION OF MOVING INTERFACE FROM SIMILARITY SOLUTION

MFE SOLUTION FOR $U=V^{1/5}$



SIMILARITY SOLUTION



9. CONCLUSION

We have shown that by applying an appropriate transformation to the original partial differential equation for non-linear diffusion (2.1) and solving for the transformed variable with MFE we may track the position of the moving interface very accurately and resolve the steep moving front to high accuracy using very few elements. The heuristic choice of penalty parameters has been avoided throughout and by controlling the size of the time step the problem of element folding has been overcome without the use of excessively small time steps, with the simple adaptive time step procedure increasing the size of the time step effectively through the computation.

Alternative methods for treating second derivative terms have been discussed and a straightforward and rigorous interpretation has been given in two dimensions. In one dimension the use of a recovery procedure for the second derivative terms together with an elementwise formulation of the method has allowed us to prove a maximum principle in this case for the gradient of the MFE approximation.

10. ACKNOWLEDGEMENTS

I would like to acknowledge the encouragement and assistance given to me throughout this work by my supervisor Dr. M.J. Baines. I am also grateful to Colin Please, Peter Sweby and Andy Wathen for many useful discussions and suggestions.

Finally I would like to acknowledge the support of the SERC through a CASE Studentship with A.W.R.E. Aldermaston.

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