D E P A R T M E N T O F M A T H E M A T I C S

MOVING FINITE ELEMENT PROCEDURES

AND

OVERTURNING SOLUTIONS

M.J. Baines and C.P. Reeves

Numerical Analysis Report 11/90

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Abstract

It is shown that the Moving Finite Element procedure remains valid for overturning solutions provided that it is written as a two stage minimisation procedure, the second stage being purely algebraic. Both global and local methods are described from this point of view, together with the details of their implementation.

§1. Introduction

The Moving Finite Element (MFE) procedure of Miller [1],[2] for time-dependent PDE's, which couples the solver with the grid motion, is based on a double minimisation of the residual norm [3]. In [4] a two-stage form of the basic procedure is presented which consists of an elementwise projection stage followed by a transfer of element information to the nodes. A description of this two-stage form in terms of function norms is given in [5]. The two-stage MFE procedure has been used in [6] to generate multi-valued solutions of nonlinear PDE's by permitting node overtaking.

It has been pointed out by Sweby [7] that, in the case of multi-valued solutions, the unweighted residual norm in the basic single-stage form of the procedure is no longer positive definite, i.e. it is no longer a norm. We show here that the two-stage form of the procedure is however still valid provided that the second stage is defined in a nodewise manner. This aspect is considered in §2 and 3 below. (The issues treated in these sections do not arise when the Gradient Weighted form of MFE is used [8], since the residual norm is always positive definite in this case. This is ensured by considering the nodal speeds normal and tangential to the solution surface.)

The local MFE procedure of Baines [9], which provides a more compact procedure for the same purpose, is also cast in a two-stage form in [4]. The corresponding function norm is again described in [5]. The discrete version of this norm, which is valid for overtaking and therefore multi-valued solutions, is discussed in §4. The flexibility of the two-stage approach is exploited in §5 where various special cases

are considered, appropriate to one- and two-dimensional versions of both moving and fixed finite elements. Implementation of these procedures is discussed in §6.

An alternative MFE procedure (splitting the solver from the grid motion) has been suggested [10] (see also [11]), and is considered from the present point of view in §7. It is also noted that in the special case of a fixed grid the local method is closely associated with mass lumping [12].

Finally in §8 the use of standard piecewise linear basis functions in one-stage forms of the three approaches is reconstructed from the two-stage approach in the non-overturning case.

§2. The Two-Stage Procedure

In this section we describe the one-and two-stage forms of the MFE procedure and characterise their norms.

Let u be a continuously differentiable function of the space variables $\underline{x}$ and time t in $\Omega \times (0,t)$, where Ω is a polygonal region $\subset \mathbb{R}^d$ (d being the dimension of the space used) and t_1 is a fixed positive time, and let $\mathcal{L}(\cdot)$ be a first order operator continuously differentiable in $\underline{x}$ such that

$$u_t - \mathcal{L}(u) = 0 \ . \tag{2.1}$$

For each t let $U \in S$ and $U_t \in T$ be finite-dimensional approximations to u and u_t which are piecewise linear functions on Π, a partition of Ω with linear facets (triangles, tetrahedra, etc.), where S and T are (generally distinct) linear spaces of such functions.

Since $\mathcal{L}(\cdot)$ is first order continuously differentiable, $\mathcal{L}(U)$ exists and is continuous in Ω except at internal boundaries of the partition Π.

Assuming that U is continuously differentiable with respect to t,

$$U_t - \mathcal{L}(U) \tag{2.2}$$

exists and is integrable over Ω for all $t \in (0, t_1)$. Although u satisfies (2.1) the function (2.2) will not vanish in general and we may define the residual

$$R = U_t - \mathcal{L}(U) \ . \tag{2.3}$$

Now define the usual L_2 inner product of square-integrable functions f and g as

$$\langle f, g \rangle = \int_{\Omega} f(\underline{x}) g(\underline{x}) W(\underline{x}) d\underline{x} \tag{2.4}$$

where $W(\underline{x})$ is a positive weight function, and the L_2 norm of g (squared) as

$$||g||^2 = \langle g, g \rangle \ . \tag{2.5}$$

With R given by (2.3) the MFE procedure minimises $||R||^2$ over all $U_t \in T$, leading to nodal velocities which provide the evolution of both grid and solution [1],[2].

Consider now the space S^* of piecewise linear discontinuous functions on the partition Π (called DPL in [5]). Following [5], we define g^* to be the L_2 projection of g into S^* (see fig. 2.1).

By definition $g^* \in S^*$ minimises

$$||g - g^*|| \tag{2.6}$$

or equivalently [13]

$$\langle g - g^*, g^* \rangle = 0 \ . \tag{2.7}$$

Then

$$||g||^2 = \langle g,g \rangle = \langle g - g^*, g - g^* + g^* \rangle$$

$$= \langle g - g^*, g - g^* \rangle + \langle g^*, g^* \rangle + 2 \langle g - g^*, g^* \rangle$$

$$= ||g - g^*||^2 + ||g^*||^2 \tag{2.8}$$

by (2.7). Thus we may write the minimisation of

$$||R||^2 = ||R - R^*||^2 + ||R^*||^2 \tag{2.9}$$

as two separate minimisations, those of $||R - R^*||$ and $||R^*||^2$, where R^* is the projection of R into S^*. This is a consequence of the orthogonality condition (2.7) and gives the two-stage method.

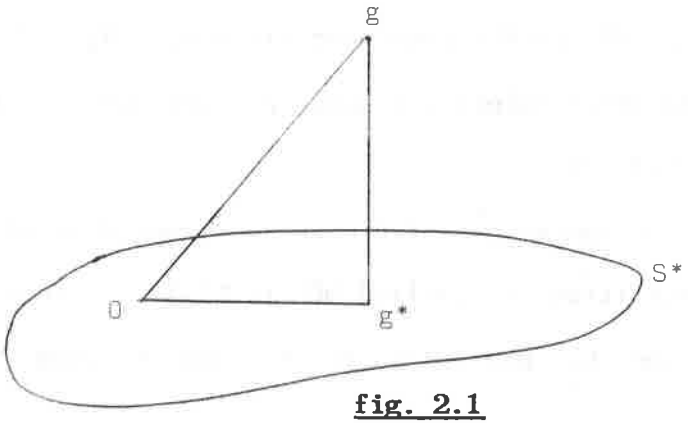

fig. 2.1

Let $\mathcal{L}(U)^*$ be the L_2 projection of $\mathcal{L}(U)$ into S^*. Then, since $U_t \in S \subseteq S^*$,

$$R - R^* = -\mathcal{L}(U) + \mathcal{L}(U)^* \tag{2.10}$$

and stage one, the minimisation of the first of (2.8) with $g = R$, $g^* = R^*$, is therefore to find $\mathcal{L}(U)^*$ (see fig. 2.2).

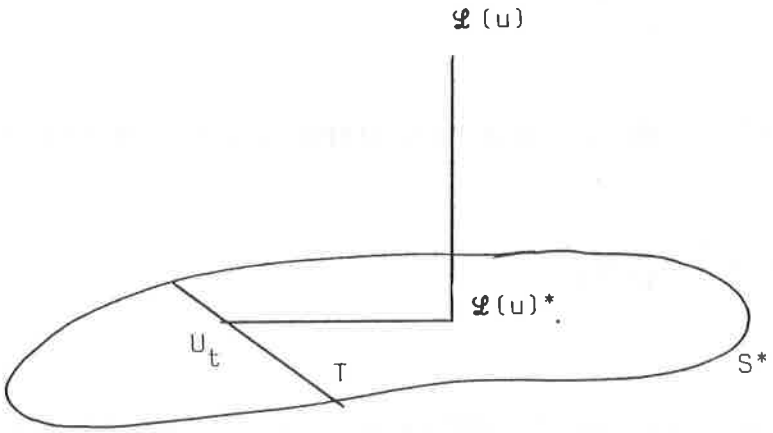

fig. 2.2

Stage two, the minimisation of the second term of (2.8) with $g^* = R^*$ is to obtain the $U_t \in T$ which minimises

$$R^* \equiv U_t - \mathcal{L}(U)^* \tag{2.11}$$

in the L_2 norm. Since $U_t \in T$, $\mathcal{L}(U)^* \in S^*$, with T, S^* both finite-dimensional spaces, we may describe the minimisation of the L_2

norm of (2.11) as the minimisation of the ℓ_2 norm of a vector of coordinates of (2.11). For this purpose we introduce sets of basis functions for the spaces T and S^*.

Let therefore $\{\phi_i\}$ and $\{\delta_i\}$ be sets of piecewise linear basis functions which span the spaces S^* and T, respectively (c.f. [5]), and let $\{w_i\}$ and $\{q_i\}$ be the corresponding sets of coefficients (coordinates) for the functions $\mathcal{L}(U)^* \in S^*$ and $U_t \in T$, so that

$$\mathcal{L}(U)^* = \sum_i w_i \phi_i \qquad\qquad U_t = \sum_i q_i \, \delta_i \, . \tag{2.12}$$

Since $T \subseteq S^*$, each δ_i may be written in terms of the $\{\phi_i\}$,

$$\delta_i = \sum_j \mu_{ij} \, \phi_j \, , \tag{2.13}$$

say, where μ_{ij} are coefficients, so that U_t may be written

$$U_t = \sum_i \sum_j q_i \, \mu_{ij} \, \phi_j \;=\; \sum_i \sum_j q_j \, \mu_{ji} \, \phi_i \, . \tag{2.14}$$

Thus, from (2.11)

$$||R^*||^2 = || \sum_i \left[\sum q_j \, \mu_{ji} - w_i \right] \phi_j ||^2 \tag{2.15}$$

which is to be minimised over the coefficients of U_t, i.e. the $\{q_i\}$.

Rewrite (2.15) as

$$||R^*||^2 = \langle \sum_i \left[\sum_j q_j \mu_{ji} - w_i\right] \phi_i \, , \sum_k \left[\sum_\ell q_\ell \mu_{\ell k} - w_k\right]\phi_k \rangle$$

$$= \sum_i \sum_k \left[\sum_j q_j \mu_{ji} - w_i\right]\left[\sum_\ell q_\ell \mu_{\ell k} - w_k\right] \langle \phi_i, \phi_k \rangle \, . \qquad (2.16)$$

This is a quadratic form in the unknown coefficients $\{q_i\}, \{q_\ell\}$ whose minimisation leads to a set of linear equations for the q's in terms of the w's, which are known through $\mathcal{L}(U)^*$ from the first stage.

Indeed, from (2.10), (2.12) the coefficients $\{w_i\}$ in the first stage minimise

$$|| -\mathcal{L}(U) + \sum w_i \, \phi_i ||^2 \, , \qquad (2.17)$$

leading to a set of linear equations for the $\{w_i\}$ in terms of $\langle \phi_i, \mathcal{L}(u) \rangle$. Once the q's have been found, U_t is known and hence the nodal motions.

Thus we may accomplish the minimisation of $||R||^2$, henceforth known as the global method, in two stages, the second of which may be written as an ℓ_2 projection. The advantage of this approach will be seen when we come to consider multi-valued solutions and the alternative local MFE method [4] in the following sections.

§3 Multi-Valued Solutions

The difficulty with multi-valued functions, which arise from overturning solutions, is that unless the quantity

$$||R||^2 \qquad (3.1)$$

is decomposed into element contributions and suitably weighted, it is in general no longer a positive definite function of U_t. However, by approaching the minimisation of (3.1) in two stages (c.f.(2.9)), via separate minimisations of

$$||R - R^*||^2 \tag{3.2}$$

(which is elementwise and always positive definite), and $||R^*||^2$ (which may be considered as an ℓ_2 projection unaffected by the sense of integration) the difficulty may be easily overcome.

In effect we simply redefine the norm to be minimised as

$$||R-R^*||^2 + \sum_i \sum_k \left[\sum_j q_j \mu_{ji} - w_i\right]\left[\sum_\ell q_\ell \mu_{\ell k} - w_k\right] < \phi_i, \phi_k > \tag{3.3}$$

(c.f. (2.16)), which reduces to $||R||^2$ in the single-valued non-overturning case. Note that it is necessary to specify the basis $\{\phi_i\}$ for S^* and also the μ_{ji} but otherwise there is no restriction.

The w's come from minimising $||R^*||^2$, as in (2.17), and the objective here is to determine the q's, which will give the nodal motions. In programming terms, however, the two-stage code is unaltered.

We now turn our attention to the compactness of the sets of linear equations for the w's and the q's, which links up with the local method of [4].

§4 Local Method

Since S^* is the space of piecewise linear discontinuous functions

we may choose $\{\phi_i\}$ to be a set of linear basis functions whose members are zero except on particular elements of Π. As a result the set of linear equations for the w's decouples into separate element-by-element sets of $d + 1$ equations, where d is the dimension of the physical space.

On the other hand, it is obvious from the double summation in (3.3) that the set of linear equations for the q's is fully coupled in general, although the matrix involved may be sparse.

With the aim of obtaining a decoupling of the set of linear equations for the q's along the lines of that for the w's, we now define, following [5], a diagonal norm for the minimisation of $||R^*||^2$.

This may be approached by introducing a new inner product $((\cdot,\cdot))$ with the property

$$((\phi_i,\phi_j)) = \begin{cases} < \phi_i, \phi_j > & i = j \\ 0 & i \neq j \end{cases} \tag{4.1}$$

and using it in (2.16), so that (2.15) and (2.16) become

$$\sum_i \sum_k \left[\sum_j q_j \mu_{ji} - w_i\right]\left[\sum_\ell q_\ell \mu_{\ell k} - w_k\right]((\phi_i,\phi_k))$$

$$= \sum_i \left[\sum_j q_j \mu_{ji} - w_i\right]^2 < \phi_i,\phi_i >$$

$$= \sum_i ||\sum_j (q_j \mu_{ji} - w_i)||_d^2 \cdot ||\phi_2||^2 = ||R^*||_{dd}^2 \tag{4.2}$$

where $||\cdot||_{dd}$ denotes a discrete diagonal ℓ_2 norm. This effectively diagonalises the quadratic form (2.16) and, if i is taken to be a sum over nodes, leads to a decoupling of the set of linear equations for the q's into separate node-by-node sets of equations.

The L_2 norm $||\cdot||_d$ associated with the ℓ_2 norm $||\cdot||_{dd}$ is given by

$$||R^*||_d^2 = ((R^*,R^*)) \tag{4.3}$$

(c.f. (2.15) and [5]) and is evidently dependent on the basis $\{\phi_i\}$ chosen (because of (4.2)).

In parallel with (3.3), then, we define the local norm (capable of handling multi-valued solutions) as

$$||R-R^*||^2 + \sum_i \sum_k \left[\sum_j q_j \mu_{ji} - w_i\right]\left[\sum_\ell q_\ell \mu_{\ell k} - w_k\right]((\phi_i,\phi_k))$$

$$= ||R-R^*||^2 + \sum_i \left[\sum_j q_j \mu_{ji} - w_i\right]^2 < \phi_i,\phi_i > . \tag{4.4}$$

Again the ϕ_i must be specified (satisfying (4.1)) and also the μ_{ji}. This norm is equal to the norm $|||R|||^2$ defined in [5] in the case of non-overturning solutions, and then the method corresponds to the two stage local method described in [4] (see fig. (4.1)). In linear algebra terms each stage of the procedure is now broken into separate compact element-by-element or node-by-node systems of equations.

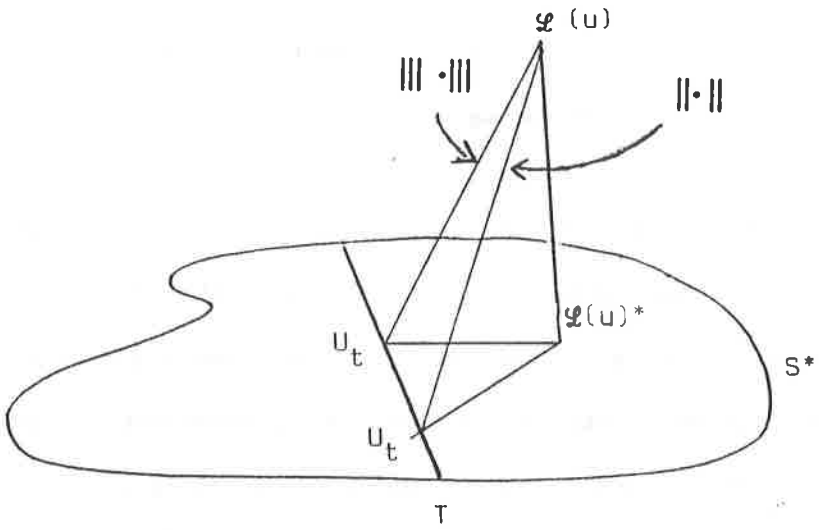

Fig. 4.1.

§5. The Spaces S^*, T and S and the Coefficients μ_{ji}

We have defined S^* to be the space of piecewise linear discontinuous finctions on the partition Π. The space T containing U_t depends on the number of physical dimensions d. If d = 1, the moving element form of U_t also lies in S^*, so that in this case $T \equiv S^*$. In that case the second stage is merely a change of coordinates (see fig. 5.1). If however d > 1, the function U_t lies in a smaller space than S^* (see fig. 2.2). In that case the second stage is a projection into the smaller space.

The fixed element form of U_t is always in S, the space chosen for U. In that case the space T is the same as the space S. A convenient basis $\{\delta_i\}$ for the space S is the set of standard linear hat functions, in which case (2.12) becomes

$$\delta_i = \sum_j \phi_{ij} \qquad (5.1)$$

elements of

the patch

so that $\mu_{ij} = \begin{cases} 1 & j \in \text{patch of elements around node } i \\ 0 & \text{otherwise} \end{cases}$. (5.2)

The result of this choice in the present approach, when minimising the L_2 norm, is the standard Galerkin weak form of the original equation (2.1), to be referred to as the Fixed Finite Element (FFE) method. (The same construction holds when the nodes have prescribed motions, the only modification needed being to $\mathcal{L}(U)$ and not to the spaces - as in the "split" method in §7 below). In the case of the $|||\cdot|||$ norm we obtain a local FFE method equivalent to a centred Petrov-Galerkin form of equation (2.1) (see [4] and §8).

Returning to the MFE method in the case $d > 1$, although $U \in S$ the time derivative U_t belongs to a larger space which includes functions which are discontinuous along element boundaries in a particular way [1],[2],[4]. The corresponding basis functions include the δ's given by (5.1) but also δ's of the form

$$\delta_i = \sum_{\substack{j \\ \text{elements} \\ \text{of the Patch}}} \left[-m_{ij}\right]\phi_{ij} \qquad (5.3)$$

(c.f. (2.13)), where the m_{ij} are the (prescribed) components of the gradient of U . In this case

$$\mu_{ij} = \left.\begin{cases} 1 \\ -m_{ij} \\ 0 \end{cases}\right\} \begin{array}{l} j \in \text{ patch of elements around node } i \\ \text{otherwise} \end{array} \qquad (5.4)$$

In the overtaking case the function U_t is replaced by a vector of nodal motions and the spaces T and S are no longer function spaces. However, we can still use the space S^* with basis $\{\delta_i\}$ and the coefficients μ_{ij} to insert into the discrete norm of (3.3) and (4.4).

§6. Implementation of the Methods

We now describe the implementation of the two-stage and one-stage methods and demonstrate how they give rise to sets of linear equations for MFE. The methods are applied to the equation

$$u_t - \mathcal{L}(u) = 0 \tag{6.1}$$

where $u = u(x,t)$ and $\mathcal{L}(u)$ contains only x,u and first derivatives of u. The methods are written down only for the 1-d case except when this is insufficiently general (i.e. the 1-D global and local methods are equivalent but the 2-D versions are not).

Implementation of the 1-D global method

For this case we describe only the implementation of the two-stage method for multivalued solutions, since the non-overturning case is the already well-documented minimisation of the L_2 norm [1],[2],[4].

Using the ideas of Miller & Carlson ([8], see also [14]), the approximation to u may be written as a linear combination of local elementwise basis functions $\phi_k^{(1)}, \phi_k^{(2)}$ (see fig. 6.1).

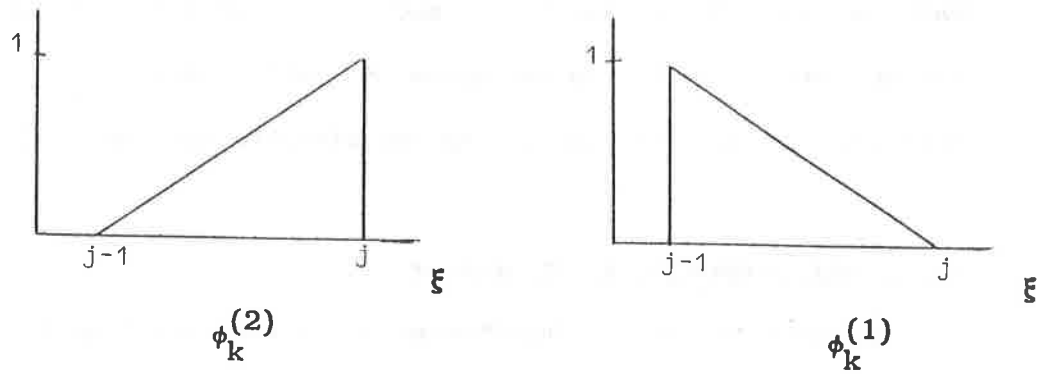

fig. 6.1

Let u be approximated by $\tilde{U}_k$ in element k where

$$\tilde{U}_k(\xi,\tau) = a_{j-1}(\tau)\phi_k^{(1)}(\xi) + a_j(\tau)\phi_k^{(2)}(\xi) \ . \qquad (6.2)$$

ξ being a reference variable and with $\tau = t$ (c.f. [10]). Also let the transformation $\xi \to x$ in element k be approximated by

$$\tilde{X}_k(\xi,\tau) = s_{j-1}(\tau)\phi_k^{(1)}(\xi) + s_j(\tau)\phi_k^{(2)}(\xi) \ . \qquad (6.3)$$

Using the chain rule, we get

$$\frac{\partial}{\partial\tau} = \frac{\partial t}{\partial\tau}\frac{\partial}{\partial t} + \frac{\partial\tilde{X}}{\partial\tau}\frac{\partial}{\partial x}$$

$$= \frac{\partial t}{\partial\tau}\frac{\partial}{\partial t} + \left\{\dot{s}_{j-1}\,\phi_k^{(1)} + \dot{s}_j\,\phi_k^{(2)}\right\}\frac{\partial}{\partial x} \qquad (6.4)$$

where the dot notation indicates differentiation with respect to τ .

Hence (6.1) becomes

$$\frac{\partial \tilde{U}_k}{\partial t} = \dot{a}_{j-1}\,\phi_k^{(1)} + \dot{a}_j\,\phi_k^{(2)} - U_x\{\dot{s}_{j-1}\,\phi_k^{(1)} + \dot{s}_j\,\phi_k^{(2)}\}$$

$$= \{\dot{a}_{j-1} - m_k\,\dot{s}_{j-1}\}\,\phi_k^{(1)} + \{\dot{a}_j - m_k\,\dot{s}_j\}\,\phi_k^{(2)} \qquad (6.5)$$

where m_k is the gradient U_x in the k^{th} element.

Alternatively, writing

$$\left.\begin{aligned} w_k^{(1)} &= \dot{a}_{j-1} - m_k\,\dot{s}_{j-1} \\ w_k^{(2)} &= \dot{a}_j - m_k\,\dot{s}_j \end{aligned}\right\} \qquad (6.6)$$

we have

$$\frac{\partial \tilde{U}_k}{\partial t} = w_k^{(1)}\,\phi_k^{(1)} + w_k^{(2)}\,\phi_k^{(2)} \,. \qquad (6.7)$$

The two forms (6.5) and (6.7) are those used by Miller and Carlson [8] and by Baines and Wathen [4], respectively.

Stage 1

We first minimise

$$\left|\left|\frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k)\right|\right| \qquad (6.8)$$

over each element k, with respect to $w_k^{(1)}$, $w_k^{(2)}$ (c.f. (2.17)) using

(6.6), obtaining the system

$$\left.\begin{aligned} \langle \phi_k^{(1)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) \rangle &= 0 \\ \langle \phi_k^{(2)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) \rangle &= 0 \end{aligned}\right\} \qquad (6.9)$$

which gives two equations in two unknowns for each element k. The system is non-singular in general and, using (6.7), can be written in the form

$$C_k \underline{w}_k = \underline{b}_k \qquad (6.10)$$

where $\underline{w}_k = \begin{bmatrix} w_k^{(1)} \\ w_k^{(2)} \end{bmatrix}$ $\qquad C_k = \frac{1}{6} \Delta s_k \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\Delta s_k = s_j - s_{j-1}$, and

$$\underline{b}_k = \begin{bmatrix} \langle \phi_k^{(1)}, \mathcal{L}(\tilde{U}_k) \rangle \\ \langle \phi_k^{(2)}, \mathcal{L}(\tilde{U}_k) \rangle \end{bmatrix} .$$

The square of the norm (6.8) can then be written as

$$\underline{w}_k^T C \underline{w}_k - 2 \underline{w}_k^T \underline{b} + ||\mathcal{L}(\tilde{U}_k)||^2 . \qquad (6.11)$$

(The $\underline{w}_k$ variables are in fact elementwise combinations of the Legendre dual variables discussed in [10]).

Alternatively, following Miller & Carlson [8],[14], we may minimise (6.7) over $\dot{a}_k^{(1)}$, $\dot{a}_k^{(2)}$, $\dot{s}_k^{(1)}$, $\dot{s}_k^{(2)}$, which leads to the double system

$$\left.\begin{aligned} &< \phi_k^{(1)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) > = 0 \\ &< \phi_k^{(2)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) > = 0 \\ &< - m_k \phi_k^{(1)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) > = 0 \\ &< - m_k \phi_k^{(2)}, \frac{\partial \tilde{U}_k}{\partial t} - \mathcal{L}(\tilde{U}_k) > = 0 \end{aligned}\right\} . \qquad (6.12)$$

This gives four equations in four unknowns for each element k. The system is singular. However, considering all elements together we find that values of $\dot{a}_j$, $\dot{s}_j$ are defined on both sides of each node, i.e. from element $k-1$ we have $\dot{a}_{k-1}^{(2)}$, $\dot{s}_{k-1}^{(2)}$ and from element k we have $\dot{a}_k^{(1)}$, $\dot{s}_k^{(1)}$ where for continuity these need to be equal. To obtain this continuity and also to enforce boundary conditions, constraints must be applied. The result is an assembly of the sets (6.12) to give a non-singular system. (In fact this approach projects $\mathcal{L}(U)$ into S^* twice, as opposed to (6.9) which projects it once.)

Following [14], the singular system (6.12) can be written as

$$E_k \dot{\underline{y}}_k = \underline{G}_k \qquad (6.13)$$

where

$$\dot{\underline{y}}_k = \begin{bmatrix} \dot{a}_k^{(1)} \\ \dot{s}_k^{(1)} \\ \dot{a}_k^{(2)} \\ \dot{s}_k^{(2)} \end{bmatrix} \quad , \qquad E_k = \frac{\Delta s_k}{6} \begin{bmatrix} 2\,\underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T \\ \underline{m}\,\underline{m}^T & 2\underline{m}\,\underline{m}^T \end{bmatrix} \tag{6.14}$$

and $$\underline{G}_k = \begin{bmatrix} < \phi_k^{(1)} \quad , \mathcal{L}(\tilde{U}_k) > \\ < \phi_k^{(2)} \quad , \mathcal{L}(\tilde{U}_k) > \\ < -m_k \phi_k^{(1)} \, , \mathcal{L}(\tilde{U}_k) > \\ < -m_k \phi_k^{(2)} \, , \mathcal{L}(\tilde{U}_k) > \end{bmatrix} . \tag{6.15}$$

The square of the norm (6.8) can then be written as

$$\dot{\underline{y}}_k^T E_k \dot{\underline{y}}_k - 2\dot{\underline{y}}_k G_k + ||\mathcal{L}(\tilde{U}_k)||^2 \tag{6.16}$$

but, as we have already stated, this gives rise to a singular system unless the constraints of stage 2 are applied.

Stage 2

In stage 2 we work only with the coordinates $\underline{w}_k$ satisfying (6.10) or $\dot{\underline{y}}_k$ satisfying (6.13).

In the case of the w's we need to implement the minimisation of (2.16) with the μ's given by (5.4) in the 1-D version. This gives the

standard form [4] of the MFE equations

$$M^T C M \dot{\underline{y}} = M^T C \underline{b} = \underline{g} \ , \tag{6.17}$$

say, where $C = \text{diag}\ \{C_k\}$, $M = \text{diag}\ \{M_j\}$, $M_j = \begin{bmatrix} 1 - m_k & \\ & 1 - m_k \end{bmatrix}$, and

$$\dot{\underline{y}} = [\ldots \dot{a}_j, \dot{s}_j, \ldots,]^T \ . \tag{6.18}$$

Returning to (6.12), and applying the continuity constraints on $\dot{a}$, $\dot{s}$, also gives rise to the standard global MFE equations, as follows

$$\text{Let } \dot{\underline{Y}}_j = \begin{bmatrix} \dot{a}^{(2)}_{k-1} \\ \dot{s}^{(2)}_{k-1} \\ \dot{a}^{(1)}_{k} \\ \dot{s}^{(1)}_{k} \end{bmatrix} = R_j \begin{bmatrix} \dot{a}_{j-1} \\ \dot{s}_{j-1} \end{bmatrix} \text{ where } R_j = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix} \ , \ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.19}$$

Then over the whole system we obtain

$$\dot{\underline{Y}} = R\, \dot{\underline{y}} \tag{6.20}$$

where

$$R^T = \begin{bmatrix} 0 0 \ \ldots & & 0 \\ \vdots & & \vdots \\ & I_2 I_2 0\ 0 & \\ & 0\ 0 I_2 I_2 & \\ 0 \ \ldots & & 0 \end{bmatrix}$$

and $\dot{\underline{Y}} = \begin{bmatrix} \cdot \\ \cdot \\ \dot{\underline{Y}}_j \\ \dot{\underline{Y}}_{j+1} \\ \cdot \\ \cdot \end{bmatrix}$.

The sum of the squares of the norms (6.8) over k may now be written

$$\dot{Y}^T E \dot{Y} - \dot{Y}^T \underline{G} = \dot{\underline{y}}^T R^T E R \dot{\underline{y}} - 2\dot{\underline{y}}^T R^T G \tag{6.21}$$

and, minimising this expression over $\dot{\underline{y}}$ yields

$$R^T E R \dot{\underline{y}} = \underline{g} \tag{6.22}$$

where

$$E = \text{diag } \{E_k\} \ , \quad \underline{g} = R^T \underline{G} \ .$$

Note that applying constraints to (6.10) is equivalent to

$$\min_{\dot{s}_j, \dot{a}_j} \left\| \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a}_j \\ \dot{s}_j \end{bmatrix} - \begin{bmatrix} \dot{a}^{(2)}_{k-1} \\ \dot{s}^{(2)}_{k-1} \\ \dot{a}^{(1)}_{k-1} \\ \dot{s}^{(1)}_{k-1} \end{bmatrix} \right\} W \right\|$$

where W is a weight function (matrix). By the appropriate choice of W , we can obtain either the global or local method. Global corresponds to $W = E^{\frac{1}{2}}$, local to $W = E_d^{\frac{1}{2}}$, where $E_d = \text{diag } \{E\}$ (see below).

It is shown in [14] that (6.22) gives rise to the usual MFE system,

since

$$R^T E R = R^T \tilde{M}^T C \tilde{M} R \tag{6.23}$$

$$= M^T C M \tag{6.24}$$

$$= A$$

where

$$E_k = \tilde{M}_k^T C_k \tilde{M}_k \ , \ \tilde{M}_k = \begin{bmatrix} 1 & -M_k & 0 & 0 \\ 0 & 0 & 1 & -M_k \end{bmatrix} \tag{6.25}$$

$$\tilde{M} = \text{diag } \{\tilde{M}_k\} \ , \ M = \tilde{M} R \ , \ E = \tilde{M}^T C \tilde{M} \ . \tag{6.26}$$

Note that the difference between the matrix decompositions (6.23) and (6.24) is essentially in the use of nodal variables (in E) rather than element variables (in C).

Implementation of the Local Method

For the two-stage global and local methods both the first and second stages are identical in 1-D, whereas in higher dimensions only the first stage is the same (see §4 and [4],[5]). In 1-D $M = \tilde{M} R$ is square and if it is also non-singular we may proceed as follows

$$R^T \tilde{M}^T C \tilde{M} R \dot{\underline{y}} = \underline{g}$$

$$C \tilde{M} R \dot{\underline{y}} = (R^T \tilde{M}^T)^{-1} \underline{g}$$

$$D \tilde{M} R \dot{\underline{y}} = D C^{-1} (R^T \tilde{M})^{-1} \underline{g}$$

$$R^T \tilde{M}^T D \tilde{M} R \dot{\underline{y}} = R^T \tilde{M}^T D C^{-1} (R^T \tilde{M}^T)^{-1} \underline{g} \tag{6.27}$$

where $D = \text{diag}\{C\}$, so we get

$$M^T D M \dot{\underline{y}} = M^T D C^{-1} M^{-1} \underline{g}$$

i.e.

$$M^T D M \dot{\underline{y}} = M^T D \underline{b} \tag{6.28}$$

which is the local method.

We now consider the local method in 2-D. In this case the term $\mathcal{L}(u)$ in (6.1) contains x,y,u and first derivatives of u with respect to x and y . For both the global and local methods we apply the same first stage to obtain

$$\left.\begin{array}{l} C \underline{w}_k = \underline{b}_k \\ \text{or} \quad E_k \dot{\underline{y}}_k = \underline{G}_k \end{array}\right\} \tag{6.29}$$

where now

$$C_k = \frac{\Delta_k}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad E_k = \frac{\Delta_k}{12}\begin{bmatrix} 2\underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T \\ \underline{m}\,\underline{m}^T & 2\,\underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T \\ \underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T & 2\,\underline{m}\,\underline{m}^T \end{bmatrix} \tag{6.30}$$

Δ_k being the area of element k , and $\underline{m} = (1 \quad -m_k \quad -n_k)^T$ where m_k is the x derivative of U in element k and n_k the y derivative. Also $\underline{b}_k = \{b_{ki}\}$ m $b_{ki} = \langle \phi_k^{(i)} , \mathcal{L}(\tilde{U}_k) \rangle$.

$$G = \begin{bmatrix} \langle \phi_k^{(1)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle \phi_k^{(2)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle \phi_k^{(3)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -m_k \phi_k^{(1)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -m_k \phi_k^{(2)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -m_k \phi_k^{(3)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -n_k \phi_k^{(1)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -n_k \phi_k^{(2)} , \mathcal{L}(\tilde{U}_k) \rangle \\ \langle -n_k \phi_k^{(3)} , \mathcal{L}(\tilde{U}_k) \rangle \end{bmatrix} . \qquad (6.31)$$

(Note that in the second of (6.29) the projection of stage one into S^* is done three times.)

For the global case the projection of $\underline{w}$ (§1-3) or the constraints on $\dot{\underline{y}}_k$ are applied as before [14]. However, for the local method we implement the minimisation of (4.2) which leads to equation (6.28). In the Miller & Carlson approach define E_{dk} to be

$$E_{dk} = \frac{\Delta_k}{12} \begin{bmatrix} 2\underline{m}\ \underline{m}^T & 0 & 0 \\ 0 & 2\underline{m}\ \underline{m}^T & 0 \\ 0 & 0 & 2\underline{m}\ \underline{m}^T \end{bmatrix} = \tilde{M}_k^T D_k \tilde{M}_k \qquad (6.32)$$

where $D_k = \text{diag}\{C_k\}$, (c.f. (6.25)), so that (6.29) may be written as

$$E_{dk}\ \dot{\underline{y}}_k = E_{dk}\ E_k^{-1}\ \underline{G}_k \ . \qquad (6.33)$$

We now apply the constraints of stage 2 as before. In 2-D these

constraints need to be applied as follows

$$\dot{a}_{j\mu} = \dot{a}_{j\nu} \ , \ \dot{x}_{j\mu} = \dot{x}_{j\nu} \ , \ \dot{y}_{j\mu} = \dot{y}_{j\nu}$$

where μ, ν refer to the elements around the node j .

Let

$$Y_j = \begin{bmatrix} \vdots \\ \dot{a}_{j\mu} \\ \dot{x}_{j\mu} \\ \dot{y}_{j\mu} \\ \vdots \end{bmatrix}_{\forall\mu} = R_j \begin{bmatrix} \dot{a}_j \\ \dot{x}_j \\ \dot{y}_j \end{bmatrix} \tag{6.34}$$

where $R_j^T = [I_3, \ I_3, \ I_3]$, $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (6.35)

$$\underline{\dot{y}}^T = (\ldots, \ \dot{a}_j, \ \dot{x}_j, \ \dot{y}_j, \ldots)$$

$$\underline{\dot{Y}}^T = \{\underline{\dot{Y}}_k^T\}$$

so that

$$\underline{\dot{Y}} = R\underline{\dot{y}} \ . \tag{6.36}$$

Then, defining $E_d = \text{diag}\ \{E_{dk}\}$, $D = \text{diag}\ \{C\}$, from (6.33) this gives the system

$$E_d \dot{\underline{Y}} = E_d E^{-1} \underline{G}$$

$$E_d R \dot{\underline{y}} = E_d E^{-1} \underline{G}$$

$$R^T E_d R \dot{\underline{y}} = R^T E_d E^{-1} \underline{G}$$

$$R^T \tilde{M}^T D \tilde{M} R \dot{\underline{y}} = R^T \tilde{M}^T D \tilde{M} E^{-1} \underline{G}$$

which is equivalent to (6.28), i.e.

$$M^T D M \dot{\underline{y}} = M^T D \underline{b} . \tag{6.37}$$

§7 The Split Method and Prescribed Model Speeds

The theory in §2-5 can also be applied to other forms of 'moving' finite element methods. Here we consider the 'split method' in 1-D proposed in [10].

Consider the equation

$$u_t + f(x,u,u_x) = 0 \tag{7.1}$$

which gives rise to the basic MFE equations

$$\langle U_t + f(x,U,U_x), \alpha \rangle = 0 \tag{7.2}$$

$$\langle U_t + f(x,U,U_x), \beta \rangle = 0 \tag{7.3}$$

where α,β are the usual basis functions. The split method replaces

the β equation (7.3) by

$$< \dot{s} - \frac{\partial f}{\partial U_x} , \alpha > = 0 \qquad (7.4)$$

which also corresponds to minimisation of

$$|| \dot{s} - \frac{\partial f}{\partial U_x} || \qquad (7.5)$$

over $\dot{s}$ (c.f.(6.4)).

However, as in §2, if we allow the solution to become multi-valued the expression $|| \dot{s} - \frac{\partial f}{\partial U_x} ||^2$ is no longer positive definite, so we are no longer minimising a norm. This prompts us to apply the previous theory by again writing the minimisation as a two-stage procedure. We shall describe the implementation for equation (7.4) since the method will be similar for equation (7.2). (There is no distinction here between the two approaches of Baines & Wathen [4] and Miller & Carlson [8],[14].)

Stage 1

We first minimise (7.5) over $\dot{s}$ in each element k, i.e.

$$\min_{\dot{s}_{j-1}, \dot{s}_j} || \dot{s} - \frac{\partial f}{\partial U_x} ||^2 \qquad (7.6)$$

where

$$\dot{s} = \dot{s}_k^{(1)} \phi_k^{(1)} + \dot{s}_k^{(2)} \phi_k^{(2)} , \qquad (7.7)$$

$\dot{s}_k^{(1)}$ $\dot{s}_k^{(2)}$ being the values of $\dot{s}$ at the lhs and rhs of the element respectively (see fig. 7.1). Note that the $\dot{s}_k$ here play the role of the w's of earlier sections.

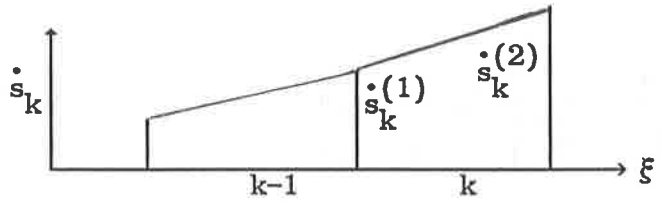

fig. 7.1.

From 7.6 we obtain the system

$$\left.\begin{aligned} \langle\, \dot{s} - \frac{\partial f}{\partial U_x} \,,\, \phi_k^{(1)} \,\rangle &= 0 \\ \langle\, \dot{s} - \frac{\partial f}{\partial U_x} \,,\, \phi_k^{(2)} \,\rangle &= 0 \end{aligned}\right\} \qquad (7.8)$$

which may be written as the overdetermined set

$$\frac{\Delta s_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{s}_k^{(1)} \\ \dot{s}_k^{(2)} \end{bmatrix} = \begin{bmatrix} \langle \frac{\partial f}{\partial u_x} \,,\, \phi_k^{(1)} \rangle \\ \langle \frac{\partial f}{\partial u_x} \,,\, \phi_k^{(2)} \rangle \end{bmatrix} \qquad (7.9)$$

(Note that (7.8) is the system which is written in §6 as

$$C_k \underline{w}_k = b_k \quad , \quad C_k = \frac{\Delta s_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad \underline{w}_k = \begin{bmatrix} \dot{s}_k^{(1)} \\ \dot{s}_k^{(2)} \end{bmatrix} \qquad (7.10)$$

and $$\underline{b}_k = \begin{bmatrix} \langle \frac{\partial f}{\partial u_x} , \phi_k^{(1)} \rangle \\ \langle \frac{\partial f}{\partial u_x} , \phi_k^{(2)} \rangle \end{bmatrix} = \begin{bmatrix} b_{1k} \\ b_{2k} \end{bmatrix}) .$$

Hence

$$\begin{bmatrix} \dot{s}_k^{(1)} \\ \dot{s}_k^{(2)} \end{bmatrix} = \frac{2}{\Delta s_k} \begin{bmatrix} 2b_{1k} - b_{1k} \\ -b_{1k} + 2b_{2k} \end{bmatrix} . \qquad (7.11)$$

However, this allows $\dot{s}_{k-1}^{(2)}$ and $\dot{s}_k^{(1)}$ to be unequal where we want a continuous solution, so constraints must be applied as in the Miller and Carlson approach of §6.

<u>Stage 2</u>

Rewriting the system (7.7) in the same form as (6.13) we get

$$E_k \dot{\underline{y}}_k = \underline{G}_k \qquad (7.12)$$

where $$E_k = \frac{\Delta s_k}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} , \quad \dot{\underline{y}}_k = \begin{bmatrix} \dot{s}_k^{(1)} \\ \dot{s}_k^{(2)} \end{bmatrix} \qquad (7.13)$$

and $$\underline{G}_k = \begin{bmatrix} b_{1k} \\ b_{2k} \end{bmatrix} .$$

Now applying the constraints gives

$$\dot{\underline{Y}}_k = \begin{bmatrix} \dot{s}_{k-1}^{(1)} \\ \dot{s}_k^{(2)} \end{bmatrix} = R_e \dot{s}_{j-1} \quad \text{where} \quad R_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \qquad (7.14)$$

So over the whole system we obtain

$$\dot{\underline{Y}} = R_o \, \dot{\underline{y}}$$

where

$$\dot{\underline{y}} = \begin{bmatrix} \vdots \\ \dot{s}_{j-1} \\ \dot{s}_j \\ \vdots \end{bmatrix}, \quad R_o = \begin{bmatrix} 10 & \cdots & 0 \\ 10 & \cdots & 0 \\ 01 & & \\ 01 & & \\ \vdots & & \\ 0 & \cdots & 1 \\ 0 & & 1 \end{bmatrix} \quad \dot{\underline{Y}} = \begin{bmatrix} \vdots \\ \dot{\underline{Y}}_j \\ \vdots \end{bmatrix} . \tag{7.15}$$

Hence as before we now solve the system

$$R_o^T \, E \, R_o \, \dot{\underline{y}} = \underline{g} . \tag{7.16}$$

where $\underline{g} = R_o^T \, \underline{G}$.

Using earlier notation this gives

$$R^T \, \tilde{M}^T \, E \, \tilde{M}_R \, \dot{\underline{y}} = \underline{g}$$

or

$$M_o^T \, C \, M_o \, \dot{\underline{y}} = \underline{g} \tag{7.17}$$

where $M_o = \tilde{M} \, R_o$.

Note that applying constraints to (7.12) is equivalent to

$$\min_{\dot{s}_j} \left\| \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{s}_j - \begin{bmatrix} \dot{s}_{k-1}^{(2)} \\ \dot{s}_k^{(1)} \end{bmatrix} \right\} W \right\|$$

where W is a weight function (matrix). Hence by application of particular W's we can obtain either a global or local 'split method': Local corresponds to $W = D^{\frac{1}{2}}$, global to $W = C^{\frac{1}{2}}$.

We now solve (7.9) to find $\dot{\underline{y}} = \begin{bmatrix} \vdots \\ \dot{s}_j \\ \vdots \end{bmatrix}$. These values of $\dot{s}_j$ may then be substituted into the system (7.2) to find the corresponding $\dot{a}_j$'s.

If we set the $\dot{s}_j$'s to be zero and then apply the above method to the $\dot{a}_j$'s, we obtain the fixed finite element method. A local fixed finite element method may also be constructed, which is given by

$$M_o^T D M_o \dot{\underline{y}} = M_o^T D C^{-1} \underline{w} \tag{7.18}$$

(c.f. (7.10)). Comparing, in 1-D

$$E_k = \frac{\Delta_k}{6} \begin{bmatrix} 2\underline{m}\,\underline{m}^T & \underline{m}\,\underline{m}^T \\ \underline{m}\,\underline{m}^T & 2\underline{m}\,\underline{m}^T \end{bmatrix} \quad \text{with} \quad D_k = \begin{bmatrix} 2\underline{m}\,\underline{m}^T & 0 \\ 0 & 2\underline{m}\,\underline{m}^T \end{bmatrix} \tag{7.19}$$

it is clear that D_k is two thirds of the lumped matrix E_k. When this lumped E_k is assembled the result is proportional to the left hand side of (7.11). Therefore the matrix in (7.11) is, apart from a factor 2/3, the lumped form of the standard fixed finite element matrix [12], where we note that the right hand side has also been lumped. The result is also true in higher dimensions but with a different factor.

§8 The One-Stage Local Method for Non-overturning Solutions

We have already considered the implementation of the following methods in two stage form, which are valid for both overturning and non overturning solutions

(a) Global MFE

(b) Local MFE

(c) Split method

i) Global

ii) Local

iii) Special case where $\dot{s} = 0$ gives FFE.

The following remarks apply to non-overturning solutions only, where one-stage methods can be used.

(a) Global MFE

The global method written as a one-stage method is the usual minimisation of the L_2 norm. A description of this may be found in e.g. [4].

(b) Local MFE

The local method may be obtained as a one-stage method using the $|||.|||$ norm of Miller [5] and its associated inner product $((.,.))$. However, to express the local method in terms of L_2 inner products use the $\tilde{\phi}$ basis functions defined below.

According to Baines & Wathen [4], Miller [5], the basis functions $\tilde{\phi}$ can be written as linear combinations of the usual ϕ basis

functions in each element and this leads to a version of the local method written as a one-stage method.

In 1-D let $\tilde{\phi}_k^{(1)}$ be of the form

$$\tilde{\phi}_k^{(1)} = a_k^{(1)}\ \phi_k^{(1)} + a_k^{(2)}\ \phi_k^{(2)} \tag{8.1}$$

where

$$\left.\begin{aligned} &< \phi_k^{(1)},\ \tilde{\phi}_k^{(1)} > = < \phi_k^{(1)},\ \phi_k^{(1)} > \\ \text{and} \\ &< \tilde{\phi}_k^{(2)},\ \tilde{\phi}_k^{(1)} > = 0 \end{aligned}\right\} . \tag{8.2}$$

This gives $\tilde{\phi}_k^{(1)} = \frac{4}{3}\ \phi_k^{(1)} - \frac{2}{3}\ \phi_k^{(2)}$ and similarly

$$\tilde{\phi}_k^{(2)} = \frac{2}{3}\ \phi_k^{(1)} - \frac{4}{3}\ \phi_k^{(2)} .$$

The same approach is valid in higher dimensions and in n dimensions it is found [5] that

$$a_1 = 1 + \frac{n}{n+2}\ ,\ a_2 = \ldots = a_n = -\frac{2}{n+2}\ . \tag{8.3}$$

Hence the $\tilde{\phi}$'s can be found in terms of the ϕ's.

The $|||.|||$ norm is defined from the inner product $((.,.))$; to express the local method in terms of the L_2 inner product we now show that

$$((g,\phi_k^{(i)})) = (g,\phi_k^{(i)})$$

over each element k .

Remembering that $|||.|||$ is defined as the sum of two stages, we first carry out the projection from L_2 space to the finite dimensional space S^*, i.e. $g \to g^*$ where $g_k^* = w_{1k}\,\phi_k^{(1)} + w_{2k}\,\phi_k^{(2)}$, $g^* = \cup g_k^*$. This gives

$$((g\ ,\ \phi_k^{(1)})) = (w_{1k}\,\phi_k^{(1)}\ ,\ \phi_k^{(1)}) + 0 + 0$$

$$= w_{1k}(\phi_k^{(1)}\ ,\ \phi_k^{(1)})\ .$$

Also by definition $g - g^*$ is orthogonal to S^* so that

$$(g\ ,\ \tilde{\phi}_k^{(1)})) = (g^*\ ,\ \tilde{\phi}_k^{(1)}) = (w_{1k}\,\phi_k^{(1)} + w_{2k}\phi_k^{(2)}) = w_{1k}(\phi_k^{(1)}\ ,\ \phi_k^{(1)})\ .$$

It follows that

$$((g\ ,\ \tilde{\phi}_k^{(1)})) = (g\ ,\ \tilde{\phi}_k^{(1)})\ . \qquad (8.4)$$

Hence writing $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ as

$$\tilde{\alpha}_k = \tilde{\phi}_k^{(1)} + \tilde{\phi}_k^{(2)}\ ,\ \tilde{\beta}_k = u_x\,\tilde{\alpha}_k \quad \text{(see fig. 8.1)} \qquad (8.5)$$

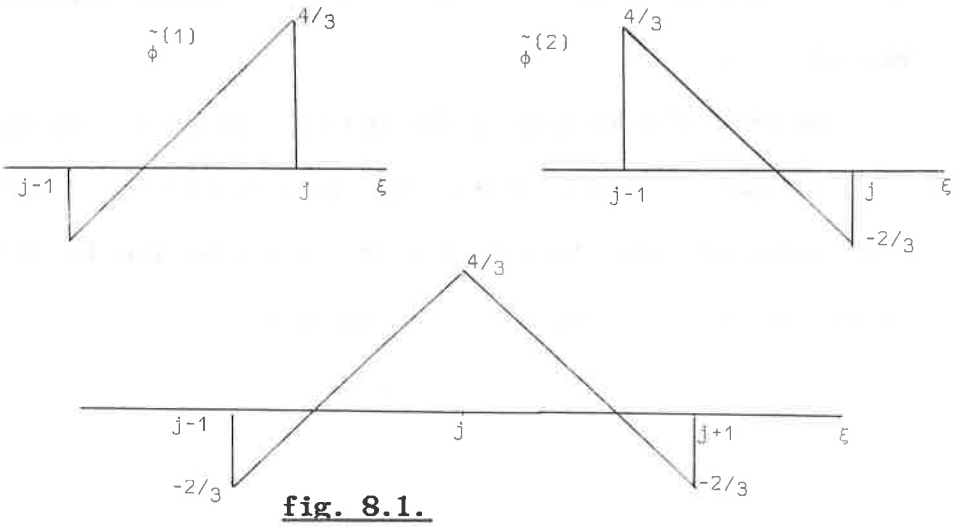

fig. 8.1.

we get the local system

$$\left.\begin{aligned} &\langle \dot{\tilde{u}}_k - L(\tilde{u}_k) \ , \ \tilde{\alpha}_k \rangle = 0 \\ &\langle \dot{\tilde{u}}_k - L(\tilde{u}_k) \ , \ \tilde{\beta}_k \rangle = 0 \end{aligned}\right\} \qquad (8.6)$$

which has been shown [4] to be the local method.

(c) **Split Method**

Since the split method may be expressed in either a global or local form it will follow either (a) or (b) above which have already been considered.

§9. Conclusion

We have shown that, by writing the MFE residual minimisation procedure in two stages and writing the second stage explicitly as an ℓ_2 projection, the procedure (without weights) is valid for overturning solutions.

The breakdown into stages throws light on the two implementations due to Baines & Wathen [4] and Miller & Carlson [8],[14] and also shows up the distinction between the standard (global) method and the local method of [4].

We have also discussed the implementation of the split method of [1], methods with other prescribed motions and the fixed method from this point of view, identifying in particular the local fixed node procedure which is equivalent to lumping.

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