Variational Principles for Free Surface Flows

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Abstract

Hamilton's principle is used to create a variational principle which has as its natural conditions the equations of irrotational motion in an incompressible, homogeneous fluid with a free surface.

By applying the shallow water approximation to the flow variables, this variational principle is reduced to one whose natural conditions are the corresponding shallow water equations of motion. For unsteady shallow water flow four functionals, whose integrands are related by Legendre transformations, are generated. Boundary terms are added to these functionals to give variational principles whose natural conditions include boundary conditions and initial conditions as well as the equations of motion. These variational principles are reduced to corresponding variational principles for steady shallow water flows by assuming that the flow variables are independent of time. The natural conditions of the variational principles derived by this method then include the steady state equations of motion in shallow water and boundary conditions on certain flow variables.

Constrained variations are made on the unsteady and steady shallow water functionals and some reciprocal variational principles are established.

1 Introduction

The purpose of this report is to identify variational principles for non-linear, irrotational, free surface flows of a homogeneous, incompressible fluid over a fixed, prescribed bed profile. The application of such principles to the computation of approximations to flows will be considered in a later report.

Luke [1] showed that a variational principle in which the integrand is taken to be the fluid pressure, as given by Bernoulli's energy integral, has as its natural conditions the equations governing a free surface flow. These are Laplace's equation for the velocity potential, in the fluid domain, the no-flow condition across the bed and the dynamic and kinematic free surface conditions. Luke did not consider conditions on lateral boundaries, such as inflow or outflow conditions, or initial conditions. His principle, which provides a starting point for this investigation, is given below in a slightly enhanced form which allows for beds of arbitrary shape (Luke considered only the case of a horizontal, flat bed).

Although Luke's principle is a valuable contribution to this area, it was presented in the form of an observation rather than being derived by some systematic means. Luke alludes to Hamilton's principle but does not forge a connection between this and his 'pressure' principle.

A recent review by Salmon [2] considers applications of classical Hamiltonian theory to fluid mechanics, which started with Clebsch [3]. Many of these applications have been within the area of gas dynamics, and general free surface flows have evidently not been considered in this context, although Miles and Salmon [4] have used Hamilton's principle to derive equations describing the motion of weakly dispersive non-linear gravity waves. There is a point of contact between free surface flows and gas flow if one invokes the shallow water approximation and we explore this connection in the report. We do not attempt to translate the full machinery of Hamiltonian mechanics into the fluid context of interest here and take a more simplistic view than Salmon. There is no doubt more to be had in the way of structural results than we seek here, as may be inferred from Salmon's review article.

To appreciate the issues involved, it is necessary to give Hamilton's principle in its classical form,

$$\delta \int_{t_1}^{t_2} L\, dt = 0, \tag{1.1}$$

where the Lagrangian $L = T - V$, T and V respectively denoting the kinetic and potential energies of the mechanical system being considered. Thus, if the i-th particle of an n-particle system has mass m_i and position $\mathbf{x}_i(t)$ at time t,

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{x}}_i . \dot{\mathbf{x}}_i,$$

$V = V(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is a given function and therefore $L = L(\mathbf{x}_1, \ldots, \mathbf{x}_n, \dot{\mathbf{x}}_1, \ldots, \dot{\mathbf{x}}_n)$. The principle (1.1) produces the usual Euler equations.

The Hamiltonian of the system is

$$H = \sum_{i=1}^{n} \mathbf{p}_i . \dot{\mathbf{x}}_i - L, \tag{1.2}$$

where $\mathbf{p}_i = \partial L / \partial \dot{\mathbf{x}}_i$ defines the conjugate momenta, the construction (1.2) being an example of a Legendre transformation, and (1.1) with (1.2) yields the canonical equations

$$\dot{\mathbf{x}}_i = \frac{\partial H}{\partial \mathbf{p}_i} \;\; , \;\; \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{x}_i} \qquad (i = 1, \ldots, n).$$

It is clear that a direct application of Hamilton's principle to a fluid flow requires the use of Lagrangian coordinates, in which the position of a fluid particle at time t is denoted by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ where $\mathbf{X}$ is the initial location of that particle. The label $\mathbf{X}$ effectively replaces the label i in the point mass system above, and the summation is correspondingly replaced by an integration over the domain occupied by the fluid at time $t = 0$.

In the context of compressible flow, Seliger and Whitham [5] have shown that this view of Hamilton's principle is correct, in that it produces the equations of momentum balance in the gas, in Lagrangian form.

From a practical point of view, the Eulerian framework is more useful than the Lagrangian system. Salmon [2] discusses the translation of Hamilton's principle from one framework to the other. Here we seek a direct way of using Hamilton's principle in the Eulerian context. The key point is that conservation of mass is implicit in the Lagrangian setting because one integrates over all the mass in the system. In Eulerian coordinates, however, where the flow through a domain fixed in space (by lateral boundaries for free surface flows) is considered, conservation of mass is not automatic and has to be enforced. One way is to augment the variational principle (1.1) by the addition of mass flow constraints, using Lagrange multipliers.

This is essentially the idea used by Seliger and Whitham [5] for compressible flows, although in their case two other constraints on the Lagrangian are evidently necessary. The energy equation, in the form of entropy conservation for a particle, is one. The other constraint is concerned with the conservation of particle identities. In the area of irrotational free surface flows, the entropy does not appear and conservation of particle labels is not significant. This is also the case in shallow water theory, where the gas dynamics analogy can be invoked.

It is shown below that 'Hamilton's principle', formed by constraining the Lagrangian $L = T - V$ by conservation of mass and by taking the 'system' to be all of the fluid particles in an assigned domain at time $t \in [t_1, t_2]$, does indeed yield the correct governing equations of unsteady, free surface flow. We follow Luke

[1] at this stage in ignoring boundary and initial conditions, and show that, with this limitation, the Eulerian version of Hamilton's principle, constructed as we have indicated, is the same as Luke's 'pressure' principle. Thus Luke's principle is merely a disguised version of Hamilton's principle.

The same interpretation of Hamilton's principle is equally successful for shallow flows, as is shown in Section 3. Moreover, we can implement the shallow water approximation in Hamilton's principle for general free surface flows, and show that the resulting principle is identical to that obtained by an *ab initio* approach to the shallow flow problem.

Alternative representations of the variational principle for shallow flow are available, based on the notion of a closed quartet of Legendre transformations introduced by Sewell [6]. In particular one version of the principle involving the pressure and another version identifiable as Hamilton's principle are connected by a Legendre transformation. These two principles are Luke's principle and Hamilton's principle for a general free surface flow, as modified by the provisions of shallow water theory. Referring to the construction (1.2), the appearance of a Legendre transformation here is not surprising.

In Section 3 we complete the shallow flow principles by including boundary terms. The modified variational principles then have, as natural conditions, boundary and initial conditions in addition to the field equations. There is some latitude in the variables which need to be assigned on lateral boundaries and initially, which is a matter of significance from a practical viewpoint. We do not explore here how to overcome one particular undesirable feature of these principles, that conditions are given on both time boundaries, $t = t_1$ and $t = t_2$, say. This is a long-standing difficulty, which is present in the classical Hamilton principle (1.1). One simple theoretical remedy is to consider only variations which vanish at $t = t_2$, but it is not clear at present how this device can be translated into an approximation method.

The latter issue is not present, of course, in steady flows which are considered in Section 4. We also give some constrained principles which lead to the notion of reciprocity, for both unsteady and steady flows. In the latter case, Bateman's [7] principles emerge. Other points of contact with existing literature are mentioned in Section 5.

This report is not intended to give an exhaustive account of all possible aspects of variational principles for free surface flows. Its main aim is to provide a clear theoretical base for the development of numerical approximation methods.

2 Variational Principles for Unsteady Free Surface Flows

In this section the equations of irrotational motion of an inviscid, incompressible, homogeneous fluid with a free surface are shown to be the natural conditions of two variational principles derived from different viewpoints. It is shown that these principles are, in fact, closely related.

Let x, y, z be cartesian coordinates, with z measured vertically upwards, and let t be the time. Consider the domain, $\tilde{\Omega}$, extending over a fixed region, D, in the x, y plane and enclosed by the surfaces $z = -h(x, y)$ and $z = \eta(x, y, t)$, where h is the known undisturbed fluid depth and η is the unknown height of the free surface above the reference level $z = 0$.

Let $\chi(x, y, z, t)$ be the velocity potential of a fluid flowing in the domain $\tilde{\Omega}$, defined such that the velocity $\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi$, and let ρ be the density of the fluid, assumed constant. Then Bernoulli's equation for the circumstances being considered may be written as

$$\frac{\tilde{p}}{\rho} + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi + gz + \chi_t = 0 \tag{2.1}$$

where g is the acceleration due to gravity, $\tilde{p} = \tilde{p}(x, y, z, t)$ is the pressure and

$$\tilde{\boldsymbol{\nabla}} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{2.2}$$

An arbitrary function of t has been absorbed into χ_t.

Luke [1] used the pressure in the form

$$-\rho\left(\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi\right)$$

as the Lagrangian density (the integrand of the Lagrangian) in a variational principle which, for the given three-dimensional domain, is

$$\delta\tilde{J}_1 = \delta\left\{\int_{t_1}^{t_2}\iint_D\int_{-h}^{\eta} -\rho\left(\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi\right)\,dz\,dx\,dy\,dt\right\} = 0 \tag{2.3}$$

where $\tilde{J}_1 = \tilde{J}_1(\eta, \chi)$. We have generalised Luke's principle slightly here by allowing for a non-constant equilibrium depth.

Let the variations in χ and η be restricted by $\delta\chi = \delta\eta = 0$ on the lateral boundary of $\tilde{\Omega}$ (that is, the boundary of D) for $t \in [t_1, t_2]$ and at times t_1, t_2 in $\tilde{\Omega}$. Then, using a mean value theorem for integrals to identify the $\delta\eta$ contribution,

$$\begin{aligned}\delta\tilde{J}_1 \;=\; & -\rho\int_{t_1}^{t_2}\iint_D\left(\delta\eta\left(\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi\right)\Big|_{z=\eta}\right.\\ & - \left(\delta\chi\left(\eta_t + \chi_x\eta_x + \chi_y\eta_y - \chi_z\right)\right)\big|_{z=\eta} - \left(\delta\chi\left(\chi_x h_x + \chi_y h_y + \chi_z\right)\right)\big|_{z=-h}\\ & \left. - \int_{-h}^{\eta}\delta\chi\,\tilde{\boldsymbol{\nabla}}^2\chi\,dz\right)dx\,dy\,dt = 0\end{aligned}$$

which yields the natural conditions

$$\begin{aligned}
\tilde{\boldsymbol{\nabla}}^2\chi &= 0 \qquad \text{in } \tilde{\Omega}, &(2.4)\\
\chi_t + gz + \frac{1}{2}\tilde{\boldsymbol{\nabla}}\chi.\tilde{\boldsymbol{\nabla}}\chi &= 0 \qquad \text{on } z=\eta, &(2.5)\\
\eta_t + \chi_x\eta_x + \chi_y\eta_y - \chi_z &= 0 \qquad \text{on } z=\eta, &(2.6)\\
\chi_x h_x + \chi_y h_y + \chi_z &= 0 \qquad \text{on } z=-h, &(2.7)
\end{aligned}$$

for $t \in [t_1, t_2]$. Equation (2.4) is the equation of conservation of mass for irrotational motion, (2.5) is the dynamic free surface condition, (2.6) is the kinematic free surface condition and (2.7) is the condition of no flow through the bed $z = -h$. Thus the variational principle (2.3) generates the governing equations of a free surface flow. We have not sought to include boundary or initial conditions as natural conditions. Notice that by using (2.1) to define the pressure in terms of χ, we are assuming both irrotationality and conservation of momentum in the flow. Put otherwise, irrotationality and conservation of momentum, in the form of the energy integral (2.1), are constraints of the variational principle.

Luke's principle can be extended to deliver irrotationality as a natural condition. The revised variational principle is

$$\delta\left\{\int_{t_1}^{t_2}\iint_D\int_{-h}^{\eta}\left(-\rho\left(\chi_t + gz + \frac{1}{2}\mathbf{u}.\mathbf{u}\right) + \tilde{\mathbf{Q}}.\left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\chi\right)\right) dz\, dx\, dy\, dt\right\} = 0 \quad (2.8)$$

where $\tilde{\mathbf{Q}} = (\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3)$ is a Lagrange multiplier and $\mathbf{u} = (u, v, w)$ is the fluid velocity. The functional in (2.8) depends on η, χ, $\mathbf{u}$ and $\tilde{\mathbf{Q}}$. The natural conditions are now given by

$$\left.\begin{aligned}
\tilde{\boldsymbol{\nabla}}.\tilde{\mathbf{Q}} &= 0\\
\tilde{\mathbf{Q}} &= \rho\mathbf{u}\\
\mathbf{u} &= \tilde{\boldsymbol{\nabla}}\chi
\end{aligned}\right\} \qquad \text{in } \tilde{\Omega}, \qquad (2.9)$$

$$\begin{aligned}
-\rho\left(\chi_t + gz + \frac{1}{2}\mathbf{u}.\mathbf{u}\right) + \tilde{\mathbf{Q}}.\left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\chi\right) &= 0 \qquad \text{on } z=\eta, &(2.10)\\
\rho\eta_t + \tilde{Q}_1\eta_x + \tilde{Q}_2\eta_y - \tilde{Q}_3 &= 0 \qquad \text{on } z=\eta, &(2.11)\\
\tilde{Q}_1 h_x + \tilde{Q}_2 h_y + \tilde{Q}_3 &= 0 \qquad \text{on } z=-h, &(2.12)
\end{aligned}$$

using the same method by which (2.4)–(2.7) were derived from (2.3). Equations (2.9) together are equivalent to (2.4); the multiplier $\tilde{\mathbf{Q}}$ is identified by $(2.9)_2$ as the three-dimensional mass flow. Using $(2.9)_2$ and $(2.9)_3$, (2.10)–(2.12) can be more clearly recognised as the dynamic free surface condition, the kinematic free surface condition and the condition of no flow through the bed respectively.

As an alternative to using the integral of the pressure as the Lagrangian in the variational principle, the difference in kinetic energy and potential energy of the flow, over the domain, may be used, that is,

$$\iint_D\int_{-h}^{\eta}\left(\frac{1}{2}\rho\mathbf{u}.\mathbf{u} - \rho g z\right) dz\, dx\, dy.$$

The condition that the functional is stationary with respect to variations is then related to Hamilton's principle in the sense described in the Introduction. Although Luke [1] mentions this more traditional form of the Lagrangian and notices that the difference between 'Hamilton's principle' and (2.3) is related to conservation of mass he does not pursue this observation.

In the fixed domain $\tilde{\Omega}$ conservation of mass,

$$\tilde{\boldsymbol{\nabla}}.\mathbf{u} = 0, \tag{2.13}$$

must be enforced (see Introduction). The kinematic free surface condition, which ensures zero mass flow across this surface, in the form

$$\left.(\eta_t + u\eta_x + v\eta_y - w)\right|_{z=\eta} = 0 \tag{2.14}$$

and the condition of zero flow through the bed,

$$\left.(uh_x + vh_y + w)\right|_{z=-h} = 0, \tag{2.15}$$

must also be enforced. These requirements are met by adding (2.13)–(2.15) to the functional under construction as constraints using the Lagrange multipliers $\lambda = \lambda(x, y, t)$, $\mu = \mu(x, y, t)$ and $\nu = \nu(x, y, z, t)$. The extra terms to be added to the functional are

$$\nu\tilde{\boldsymbol{\nabla}}.\mathbf{u},$$

integrated over $\tilde{\Omega}$, and

$$\lambda\left.(\eta_t + u\eta_x + v\eta_y - w)\right|_{z=\eta} + \mu\left.(uh_x + vh_y + w)\right|_{z=-h},$$

integrated over D. The variational principle based on Hamilton's principle is therefore

$$\begin{aligned}\delta\tilde{J}_2 &= \delta\left\{\int_{t_1}^{t_2}\iint_D\left(\lambda\left.(\eta_t + u\eta_x + v\eta_y - w)\right|_{z=\eta} + \mu\left.(uh_x + vh_y + w)\right|_{z=-h}\right.\right.\\ &\left.\left.+\int_{-h}^{\eta}\left(\rho\left(\frac{1}{2}\mathbf{u}.\mathbf{u} - gz\right) + \nu\tilde{\boldsymbol{\nabla}}.\mathbf{u}\right)dz\right)dx\,dy\,dt\right\} = 0 \end{aligned} \tag{2.16}$$

where $\tilde{J}_2 = \tilde{J}_2(\eta, \mathbf{u}, \lambda, \mu, \nu)$. Assuming that the variations in $\mathbf{u}$ and η are such that $\delta\mathbf{u} = \mathbf{0}$, $\delta\eta = 0$ on the lateral and time boundaries, as in the case of (2.3), then the natural conditions of (2.16) are given by

$$\left.\begin{aligned}\rho\mathbf{u} - \tilde{\boldsymbol{\nabla}}\nu &= \mathbf{0}\\ \tilde{\boldsymbol{\nabla}}.\mathbf{u} &= 0\end{aligned}\right\} \quad \text{in } \tilde{\Omega}, \tag{2.17}$$

$$\lambda_t + \tilde{\boldsymbol{\nabla}}.(\lambda\mathbf{u}) - \rho\left(\frac{1}{2}\mathbf{u}.\mathbf{u} - gz\right) - \mu\tilde{\boldsymbol{\nabla}}.\mathbf{u} = 0 \quad \text{on } z = \eta, \tag{2.18}$$

$$\eta_t + u\eta_x + v\eta_y - w = 0 \quad \text{on } z = \eta, \tag{2.19}$$

$$uh_x + vh_y + w = 0 \quad \text{on } z = -h, \tag{2.20}$$

for $t \in [t_1, t_2]$. The fluid is homogeneous by hypothesis, therefore if $\lambda = \rho\chi|_{z=\eta}$ and $\nu = \rho\chi$, where χ is the velocity potential, (2.17)–(2.20) together are equivalent to (2.9)–(2.12) and hence to (2.4)–(2.7).

Equation (2.16), with λ and ν as defined above and $\mu = \rho\chi|_{z=-h}$, can be shown to be the same as equation (2.8) except for terms on the boundary of D and at t_1 and t_2. The variational principle (2.16), with λ, ν and μ as stated, can be rearranged to give

$$\delta \left\{ \int_{t_1}^{t_2} \iint_D \Big(\left(\rho\chi\left(\eta_t + u\eta_x + v\eta_y - w\right)\right)\big|_{z=\eta} + \left(\rho\chi\left(uh_x + vh_y + w\right)\right)\big|_{z=-h} \right.$$
$$\left. - \int_{-h}^{\eta} \rho \left(gz + \frac{1}{2}\mathbf{u}.\mathbf{u} + \mathbf{u}.\left(\tilde{\boldsymbol{\nabla}}\chi - \mathbf{u}\right) - \tilde{\boldsymbol{\nabla}}.\left(\chi\mathbf{u}\right)\right) dz \Big)\, dx\, dy\, dt \right\} = 0.$$

Now, using the divergence theorem,

$$I = \int_{t_1}^{t_2} \iint_D \int_{-h}^{\eta} \rho\tilde{\boldsymbol{\nabla}}.\left(\chi\mathbf{u}\right) dz\, dx\, dy\, dt = \int_{t_1}^{t_2} \iint_\sigma \rho\chi\mathbf{u}.\mathbf{n}\, d\sigma\, dt$$

where σ is the whole boundary of $\tilde{\Omega}$. The parts of σ of interest here are the surfaces $z = \eta$ and $z = -h$, which contribute to I the terms

$$\int_{t_1}^{t_2} \iint_D \left(-\left(\rho\chi\left(u\eta_x + v\eta_y - w\right)\right)\big|_{z=\eta} - \left(\rho\chi\left(uh_x + vh_y + w\right)\right)\big|_{z=-h} \right) dx\, dy\, dt.$$

It follows that if Σ denotes the boundary of D and $\tilde{\mathbf{Q}}$ is defined by $(2.9)_2$ then (2.16) becomes

$$\delta \left\{ \int_{t_1}^{t_2} \left(\iint_D \int_{-h}^{\eta} \left(-\rho\left(\chi_t + gz + \frac{1}{2}\mathbf{u}.\mathbf{u}\right) + \tilde{\mathbf{Q}}.\left(\mathbf{u} - \tilde{\boldsymbol{\nabla}}\chi\right)\right) dz\, dx\, dy \right.\right.$$
$$\left.\left. + \iint_\Sigma \rho\chi\mathbf{u}.\mathbf{n}\, d\sigma \right) dt + \iint_D \left[\int_{-h}^{\eta} \rho\chi\, dz \right]_{t_1}^{t_2} dx\, dy \right\} = 0$$

which is just (2.8) with added boundary terms. The boundary terms may be neglected since variations are assumed to vanish on Σ and at times t_1 and t_2. Moreover, if (2.16) is constrained to satisfy irrotationality by substituting $\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi$ into the integrand, the resulting principle can be shown to be the same as Luke's principle, (2.3), to within boundary terms.

We have therefore established that the two variational principles — (2.8) derived from an expression for the pressure and (2.16) derived from Hamilton's principle — whose natural conditions are the equations of motion for a fluid with a free surface are the same if boundary terms are ignored. It will be shown in Section 3 that by making the shallow water approximation in (2.8) and (2.16) two further variational principles can be generated whose natural conditions are the shallow water equations of motion.

3 Variational Principles for Unsteady Shallow Water Flows

Shallow water theory offers an approximation to free surface flows in circumstances where the water depth is much less than some other characteristic length scale of the motion, such as the radius of curvature of the surface. To lowest order, this theory can be generated by assuming that the fluid pressure is hydrostatic. That is,

$$\tilde{p}(x, y, z, t) = \rho g(\eta - z), \tag{3.1}$$

taking the assumed constant surface pressure to be zero as a matter of convenience. The hypothesis (3.1) implies that the horizontal velocity components, u and v, are independent of z and that the vertical velocity component w is negligible compared with u and v. This can be summarised as

$$u_z = 0\ ,\ \ v_z = 0\ ,\ \ w = 0\,. \tag{3.2}$$

Details may be found in, for example, Stoker [8].

We can use (3.1) to determine the vertically averaged pressure $p = p(x, y, t)$ defined by

$$p = \frac{1}{\rho}\int_{-h}^{\eta} \tilde{p}\, dz,$$

from which it follows that

$$p = \frac{1}{2} g d^2, \tag{3.3}$$

where $d(x, y, t)$ is the fluid depth at the location (x, y) and at time t; that is, $d = \eta + h$. The averaging replaces the fluid motion by a representative motion in the spatial coordinates x and y in which each 'particle' can be thought of as the aggregate of all the actual fluid particles lying in the same vertical line. Stoker [8] gives the equations of irrotational motion of the reduced problem in the form

$$\begin{aligned} d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \\ \mathbf{v}_t + \boldsymbol{\nabla} E &= g\boldsymbol{\nabla} h \\ \mathbf{v} &= \boldsymbol{\nabla}\chi. \end{aligned} \tag{3.4}$$

The variables $\mathbf{Q}$ and E are given by

$$\begin{aligned} \mathbf{Q} &= d\mathbf{v} \\ E &= gd + \tfrac{1}{2}\mathbf{v}.\mathbf{v} \end{aligned} \tag{3.5}$$

where $\mathbf{v} = (u, v)$ is the velocity of the reduced problem. The variable $\mathbf{Q}$ may be called the mass flow vector since d plays the part of a density, as we indicate below, E is an energy of the flow and we have used

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Note that $(3.4)_2$, the momentum balance for the reduced problem, can be written as

$$\mathbf{v}_t + (\mathbf{v}.\boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{d}\boldsymbol{\nabla} p + g\boldsymbol{\nabla} h,$$

since $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$ and by using (3.3). This form, together with the associated mass balance equation and (3.3) shows that the shallow water equations can be regarded as the equations for a two-dimensional, compressible flow in which d plays the part of density and $g\boldsymbol{\nabla} h$ is a body force. Equation (3.3) is an 'adiabatic law'. This connection is generally referred to as the gas dynamics analogy.

Variational principles for shallow flow can be considered from two points of view. We can specialise the principles of Section 2 by applying the shallow water approximation there or we can derive principles from the equations which govern the two-dimensional compressible flow indicated above.

3.1 Shallow Water Principles from Free Surface Principles

Returning to the variational principles of Section 2, if the shallow water approximation is applied to (2.8) and (2.16) it generates functionals whose natural conditions include the equations of motion in shallow water.

Consider (2.8) — the 'pressure' principle. The effect of applying the shallow water approximation is that the potential reduces to a function $\chi = \chi(x, y, t)$ and $\mathbf{u} = \tilde{\boldsymbol{\nabla}}\chi$ is replaced by $\mathbf{v} = \boldsymbol{\nabla}\chi$. The integration with respect to z can be carried out giving the functional $J_1(\mathbf{Q}, d, \mathbf{v}, \chi)$ defined by

$$J_1 = \int_{t_1}^{t_2}\iint_D \rho\left(\frac{1}{2}gd^2 - d\left(\chi_t + \frac{1}{2}\mathbf{v}.\mathbf{v} + gd - gh\right) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\chi\right)\right)\,dx\,dy\,dt. \tag{3.6}$$

Assuming that variations vanish on the space and time boundaries the natural conditions of

$$\delta J_1 = 0$$

are

$$\left.\begin{array}{rcl} \chi_t + g\eta + \frac{1}{2}\mathbf{v}.\mathbf{v} & = & 0 \\ \mathbf{v} - \boldsymbol{\nabla}\chi & = & \mathbf{0} \\ \mathbf{Q} - d\mathbf{v} & = & \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} & = & 0 \end{array}\right\} \quad \text{in } \Omega, \tag{3.7}$$

where $\Omega = D \cup [t_1, t_2]$. These equations are respectively the integrated version of the conservation of momentum equation $(3.4)_2$, the irrotationality condition $(3.4)_3$, the definition of mass flow and the conservation of mass equation $(3.4)_1$.

Now, consider (2.16) — the 'Hamilton' principle. Under the conditions of the shallow water approximation the integral over z can be evaluated. Since, from (3.2), u and v take the same values at $z = \eta$ and $z = -h$ we can combine the terms evaluated at these levels. The result is the functional $J_2 = J_2(d, \mathbf{v}, \chi)$ given by

$$J_2 = \int_{t_1}^{t_2} \iint_D \rho \left(\frac{1}{2} d\mathbf{v}.\mathbf{v} - \frac{1}{2} g d^2 + \chi \left(d_t + \boldsymbol{\nabla}.\left(d\mathbf{v} \right) \right) + g d h \right) dx\, dy\, dt. \tag{3.8}$$

Assuming that variations vanish on the space and time boundaries then the natural conditions of

$$\delta J_2 = 0$$

are

$$\left. \begin{aligned} \chi_t + g\eta + \mathbf{v}.\boldsymbol{\nabla}\chi - \tfrac{1}{2}\mathbf{v}.\mathbf{v} &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\chi &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\left(d\mathbf{v} \right) &= 0 \end{aligned} \right\} \quad \text{in } \Omega, \tag{3.9}$$

which together are equivalent to (3.7).

Thus the ‘pressure’ and ‘Hamilton’ free surface principles reduce to ‘shallow water’ principles when the variables are approximated using shallow water theory.

The previous work is valid for any domain Ω which does not cause the fluid motion to violate the conditions of the shallow water approximation. We now consider a particular subset of such domains — those with constant undisturbed depth h.

By introducing the new potential $\phi = \chi - ght$, for which $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\chi$ and $\phi_t = \chi_t - gh$, we see from (3.6) that the modified form of J_1 in the case of constant h is

$$J_1 = \int_{t_1}^{t_2} \iint_D \rho \left(\frac{1}{2} g d^2 - d \left(\phi_t + g d + \frac{1}{2} \mathbf{v}.\mathbf{v} \right) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi \right) \right) dx\, dy\, dt. \tag{3.10}$$

As a check it can be verified that $\delta J_1 = 0$ implies

$$\left. \begin{aligned} \phi_t + g d + \tfrac{1}{2}\mathbf{v}.\mathbf{v} &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \mathbf{Q} - d\mathbf{v} &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned} \right\} \quad \text{in } \Omega. \tag{3.11}$$

These are the equations of motion, the irrotationality condition and the definition of mass flow for a domain with constant h where the function ϕ is now regarded as the potential associated with the motion. The effect of using ϕ as the potential instead of χ is simply to move the reference level from $z = 0$ to $z = -h$. The height of the free surface above the reference level is now the total fluid depth d. Using $(3.11)_{1,2}$ the integrand of (3.10) can be recognised as being of the special form of a ‘pressure’ function to which are added multiples of the integrated conservation of momentum equation and the irrotationality condition.

A corresponding rearrangement of (3.8) is possible in which χ is replaced by the new potential ϕ. Note that if h is constant

$$\begin{aligned}
&\int_{t_1}^{t_2} \iint_D \rho\left(ght\left(d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right)\right) + gdh\right)\, dx\, dy\, dt \\
&\quad = \int_{t_1}^{t_2} \iint_D \rho\left(\left(ghtd\right)_t + \boldsymbol{\nabla}.\left(ghtd\mathbf{v}\right)\right)\, dx\, dy\, dt \\
&\quad = \iint_D \rho\left[ghtd\right]_{t_1}^{t_2}\, dx\, dy + \int_{t_1}^{t_2} \int_\Sigma \rho ghtd\mathbf{v}.\mathbf{n}\, d\Sigma\, dt,
\end{aligned} \tag{3.12}$$

using the divergence theorem. Therefore, replacing χ by $\phi + ght$ in (3.8) we obtain, for constant h,

$$J_2 = \int_{t_1}^{t_2} \iint_D \rho\left(\frac{1}{2}d\mathbf{v}.\mathbf{v} - \frac{1}{2}gd^2 + \phi\left(d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right)\right)\right)\, dx\, dy\, dt. \tag{3.13}$$

We have ignored the boundary terms (that is, the terms evaluated on Σ and at t_1 and t_2), consistent with our assumption that all variations vanish at the boundaries.

The integrand of (3.13) has a similar structure to that of (3.10) — it consists of a function plus a multiple of a conservation law — this time conservation of mass.

The natural conditions of the first variation of (3.13) for variations which vanish at the boundaries are

$$\left.\begin{aligned}
\phi_t + gd + \mathbf{v}.\boldsymbol{\nabla}\phi - \tfrac{1}{2}\mathbf{v}.\mathbf{v} &= 0 \\
\mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\
d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right) &= 0
\end{aligned}\right\} \quad \text{in } \Omega, \tag{3.14}$$

which together are the same as (3.11) — the equations of motion in a domain where the undisturbed fluid depth is a constant.

The functionals (3.10) and (3.13) will be used, in Subsection 3.4, to generate variational principles whose natural conditions also include boundary conditions of the flow. First we show that functionals similar to (3.10) and (3.13) can be created by making the shallow water approximation and then generating functionals.

3.2 Shallow Water Functionals from First Principles

We now consider the application of Hamilton's principle directly to shallow flow.

As already noted, shallow flow may be regarded as a compressible flow in which d has the role of density. Pursuing this gas dynamics analogy, we find that the associated internal energy function is $\epsilon(\nu, S) = g/2\nu$ where $\nu = 1/d$ is the specific density and the entropy, S, is absent. It follows that the pressure $p(\nu, S) = -\epsilon_\nu = g/2\nu^2$ which has the values of $gd^2/2$, and that the enthalpy

$H(p, S) = \epsilon + p\nu = (2gp)^{\frac{1}{2}}$ is equal to gd in value. It is now evident that the quantity E introduced in $(3.5)_2$ is equal to $H + v^2/2$, and that the difference between the kinetic energy and potential energy of a fluid particle, $dv^2/2 - gd^2/2$, is the required Lagrangian density.

For the fixed domain considered conservation of mass must be imposed as a constraint — this notion has already been used in the derivation of the free surface 'Hamilton' principle of Section 2. Let ϕ be the Lagrange multiplier. Then this construction gives the functional

$$\int_{t_1}^{t_2} \iint_D \left(\frac{1}{2}d\mathbf{v}.\mathbf{v} - \frac{1}{2}gd^2 + \phi\left(d_t + \boldsymbol{\nabla}.\left(d\mathbf{v}\right)\right)\right)\, dx\, dy\, dt \tag{3.15}$$

which, except for the factor ρ, a constant by hypothesis, is identical to (3.13).

Ignoring boundary terms and using integration by parts and the divergence theorem, (3.15) can be rearranged to give the functional

$$\int_{t_1}^{t_2} \iint_D \left(\frac{1}{2}gd^2 - d\left(\phi_t + \frac{1}{2}\mathbf{v}.\mathbf{v} + gd\right) + d\mathbf{v}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right)\right)\, dx\, dy\, dt \tag{3.16}$$

which, except for the constant ρ in (3.10), is identical to (3.10) when the mass flow $\mathbf{Q}$ is given by $(3.5)_1$.

Thus the same basic functionals are found by making the shallow water approximation and generating functionals using Hamilton's principle as by generating functionals and then making the shallow water approximation. This derivation also shows that (3.10) and (3.13) are the same to within boundary terms after rearranging and applying the divergence theorem. The free surface 'pressure' and 'Hamilton' functionals in Section 2 were also shown to exhibit this property.

3.3 Further Principles

Before considering variational principles with natural boundary conditions we will modify slightly the 'pressure' and 'Hamilton' functionals for unsteady shallow water motion. The variational principles with boundary terms, generated using the modified functionals, can then be related to two further variational principles.

First consider (3.10) which, it will be recalled, is derived from the free surface principle (2.8) based on an expression for the pressure. In comparison with (3.16) — a rearrangement of Hamilton's principle applied to shallow water flow — we noted that the constant ρ was absent in (3.16) and now conclude that it may be set equal to unity without losing generality. As a further simplification of notation, conservation of momentum in (3.10) can be written concisely as

$$E + \phi_t = 0,$$

using $(3.5)_2$. This suggests the use of the energy E as a new variable. From $(3.5)_2$ we have

$$d = \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right)$$

which allows the definition of a new function $p(\mathbf{v}, E)$ by substituting for d in the pressure $gd^2/2$, namely

$$p(\mathbf{v}, E) = \frac{1}{2g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right)^2,$$

which has the values of pressure.

The integrand of the functional being constructed is now

$$p(\mathbf{v}, E) - d\,(E + \phi_t) + \mathbf{Q}.\,(\mathbf{v} - \boldsymbol{\nabla}\phi)\,, \tag{3.17}$$

which is the integrand of (3.10) modified in the manner outlined above. For the remainder of this report the functional to be referred to as the 'p' functional for unsteady shallow water flow will have integrand (3.17).

The 'Hamilton' principle (3.13) is modified similarly. The density ρ is taken to be unity as before, the change of variable made here is from $\mathbf{v}$ to $\mathbf{Q}$ using a rearrangement of $(3.5)_1$, $\mathbf{v} = \mathbf{Q}/d$, and the function $r(\mathbf{Q}, d)$ is defined to be

$$r(\mathbf{Q}, d) = \frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d} - \frac{1}{2}gd^2.$$

Modification of the integrand of (3.13) thus gives the expression

$$r(\mathbf{Q}, d) + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\,.$$

For the remainder of this report the functional referred to as the 'r' functional for unsteady shallow water will be the one with this term in the integrand.

The equations (3.5) have been used to change variables in the integrands of (3.10) and (3.13); the modified integrands will now be used to generate variational principles which also have natural boundary conditions.

3.4 Boundary Conditions

The boundary and initial conditions of the fluid motion have so far been ignored but will now be considered in the context of shallow water theory. The generation of the functionals (3.10) and (3.13) from the free surface principles and their subsequent modification in Subsection 3.3 has provided the base on which to form further variational principles having boundary conditions and initial conditions as natural conditions. The procedure followed now is to add boundary terms to the functionals and allow non-zero variations on the space and time boundaries.

First consider the functional with the integrand (3.17) — the 'p' functional. If we examine the associated variational principle and allow variations which do not vanish at the boundaries, the variables $\mathbf{Q}$ and ϕ will appear in space boundary terms and d and ϕ will appear in time boundary terms. With this motivation the boundary Σ of domain D is split into two parts $\Sigma = \Sigma_\phi + \Sigma_Q$. Boundary conditions for ϕ are sought on Σ_ϕ and for $\mathbf{Q}$ on Σ_Q for $t \in [t_1, t_2]$. Similarly the domain is divided into two by $D = D_d + D_\phi$ and conditions at the time boundaries t_1 and t_2 are sought for d in D_d and for ϕ in D_ϕ.

By this method a new functional is constructed — one which includes boundary terms. Let the functional $I_1(E, \mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$\begin{aligned}
I_1 = &\int_{t_1}^{t_2} \iint_D \left(p(\mathbf{v}, E) - d\,(E + \phi_t) + \mathbf{Q}.\,(\mathbf{v} - \boldsymbol{\nabla}\phi)\right)\, dx\, dy\, dt \\
&+ \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\
&+ \iint_{D_\phi} \left((d\,(\phi - h_2))|_{t_2} - (d\,(\phi - h_1))|_{t_1} \right) dx\, dy \\
&+ \iint_{D_d} \left(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \right) dx\, dy
\end{aligned} \tag{3.18}$$

where $f = f(x, y, t)$, $C = C(x, y, t)$ are given functions on Σ_ϕ and Σ_Q respectively and $g_i = g_i(x, y)$, $h_i = h_i(x, y)$ $(i = 1, 2)$ are given functions on D_d and D_ϕ respectively.

The natural conditions of the revised 'p' principle

$$\delta I_1 = 0$$

are

$$\left.\begin{aligned}
p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\
p_E - d &= 0 \\
d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \\
E + \phi_t &= 0 \\
\mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0}
\end{aligned}\right\} \qquad \text{in } \Omega,$$

$$\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 &&\text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\
\phi - f &= 0 &&\text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\
d|_{t_i} - g_i &= 0 &&\text{in } D_d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 &&\text{in } D_\phi \text{ for } i = 1, 2,
\end{aligned}$$

where

$$p_{\mathbf{v}} \equiv \frac{\partial p}{\partial \mathbf{v}} = -\frac{\mathbf{v}}{\mathbf{g}}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) \quad \text{and} \quad p_E \equiv \frac{\partial p}{\partial E} = \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right).$$

Thus the first two conditions in the domain, D, are

$$-\frac{\mathbf{v}}{\mathbf{g}}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) + \mathbf{Q} = \mathbf{0} \quad \text{and} \quad \frac{1}{g}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) - d = 0,$$

which together give

$$\mathbf{Q} = d\mathbf{v} \quad \text{and} \quad E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v},$$

so that the last three natural conditions in Ω are the conservation laws and the irrotationality condition.

Consider now the 'r' principle. The domain and domain boundary are again divided into two, as for the 'p' principle, to provide a choice of boundary and initial conditions. Using the same functions, $f = f(x,y,t)$, $C = C(x,y,t)$, $g_i = g_i(x,y)$, $h_i = h_i(x,y)$ $(i = 1,2)$, a second functional $I_2(\mathbf{Q}, d, \phi)$ can be derived, namely,

$$\begin{aligned} I_2 = & \int_{t_1}^{t_2} \iint_D \left(r(\mathbf{Q}, d) + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\; d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & - \iint_{D_d} \left((\phi\,(d - g_2))\big|_{t_2} - (\phi\,(d - g_1))\big|_{t_1} \right)\, dx\, dy \\ & - \iint_{D_\phi} \left(d\big|_{t_2} h_2 - d\big|_{t_1} h_1 \right)\, dx\, dy, \end{aligned} \tag{3.19}$$

where $\mathbf{Q}$ denotes the mass flow $d\mathbf{v}$.

The natural conditions of this 'r' principle

$$\delta I_2 = 0$$

are

$$\left. \begin{aligned} r_d - \phi_t &= 0 \\ r_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned} \right\} \qquad \text{in } \Omega,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\ \phi - f &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\ d\big|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1,2, \\ \phi\big|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1,2. \end{aligned}$$

The first two natural conditions in the domain may be rewritten as

$$-\frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d^2} - gd - \phi_t = 0 \ , \quad \frac{\mathbf{Q}}{d} - \boldsymbol{\nabla}\phi = \mathbf{0}$$

so that, using $(3.5)_1$, the equations of motion and the irrotationality condition have again been derived.

Thus there exist two functionals, (3.18) and (3.19), whose natural conditions of the first variation are the equations of motion and the irrotationality condition in the domain of the problem together with prescribed conditions on mass flow and velocity potential on the boundary of the domain, and conditions on the depth and velocity potential over regions of the domain at the initial and final times. These last conditions are not very desirable in a practical sense since they imply knowledge of the solution at the final time. In this report however we are dealing only with the derivation of variational principles and so do not investigate their practical implementation further.

3.5 A Quartet of Functionals

A sequence of Legendre transformations can be used to generate a quartet of functionals which have as natural conditions of their first variations the unsteady shallow water equations. Two such functionals — based on the p and r functions — have already been described and were independently derived. Two further functionals are now sought.

By applying the divergence theorem and integration by parts, the 'p' functional (3.18) can be expressed in the form

$$\begin{aligned} I_1 = & \int_{t_1}^{t_2} \iint_D \left(p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v} + \phi\left(d_t + \boldsymbol{\nabla}.\mathbf{Q}\right)\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\left(C - \mathbf{Q}.\mathbf{n}\right)\, d\Sigma\, dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & - \iint_{D_d} \left(\left(\phi\left(d - g_2\right)\right)\big|_{t_2} - \left(\phi\left(d - g_1\right)\right)\big|_{t_1} \right)\, dx\, dy \\ & - \iint_{D_\phi} \left(d\big|_{t_2} h_2 - d\big|_{t_1} h_1 \right)\, dx\, dy. \end{aligned} \tag{3.20}$$

Comparing this with I_2, as given by (3.19), suggests that there is a relationship between the two functions $p(\mathbf{v}, E)$ and $r(\mathbf{Q}, d)$ such that

$$r(\mathbf{Q}, d) = p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v}$$

in value, which can be confirmed directly using (3.5). The relation is in fact a Legendre transformation as we will now show.

The Legendre transform $R(\mathbf{v}, d)$ of $p(\mathbf{v}, E)$, with E and d as dual active variables and $\mathbf{v}$ passive, is defined by

$$R(\mathbf{v}, d) = Ed - p(\mathbf{v}, E) \tag{3.21}$$

and is such that

$$R_{\mathbf{v}} = -p_{\mathbf{v}} \quad , \quad R_d = E.$$

Using $(3.5)_2$ R can be constructed from (3.21) and is

$$R(\mathbf{v}, d) = \frac{1}{2} g d^2 + \frac{1}{2} d\mathbf{v}.\mathbf{v}.$$

Notice that R is equal to the total energy of a fluid particle. The function R is also a Legendre transform of $r(\mathbf{Q}, d)$, with $\mathbf{Q}$ active and d passive, in that, using $(3.5)_1$, we may write

$$R(\mathbf{v}, d) = \mathbf{Q}.\mathbf{v} - r(\mathbf{Q}, d) \tag{3.22}$$

having first derivatives

$$R_d = -r_d \quad , \quad R_{\mathbf{v}} = \mathbf{Q}.$$

This implies the required connection, that

$$r(\mathbf{Q}, d) = \mathbf{Q}.\mathbf{v} - R(\mathbf{v}, d) = p(\mathbf{v}, E) - Ed + \mathbf{Q}.\mathbf{v}$$

in value.

We can of course bypass the intermediate function R and connect p and r directly by a Legendre transformation. Since $p_{\mathbf{v}} = -\mathbf{Q}$ and $p_E = d$, then if $\mathbf{v}$ and E are both active variables, the transformation of p is

$$r(\mathbf{Q}, d) = \mathbf{Q}.\mathbf{v} - Ed + p(\mathbf{v}, E)$$

and

$$r_{\mathbf{Q}} = \mathbf{v} \ \ , \ \ r_d = -E.$$

A fourth function $P(\mathbf{Q}, E)$ completes a closed quartet of functions related by Legendre transformations and is derivable from p, r and R by using appropriate active variables. P cannot be given explicitly, but is defined by eliminating $\mathbf{v}$ and d from

$$P(\mathbf{Q}, E) = \frac{1}{2}gd^2 + d\mathbf{v}.\mathbf{v} \ \ , \ \ \mathbf{Q} = d\mathbf{v} \ \ , \ \ E = gd + \frac{1}{2}\mathbf{v}.\mathbf{v}.$$

The function P is related to p and r by

$$p(\mathbf{v}, E) - P(\mathbf{Q}, E) = -\mathbf{Q}.\mathbf{v} \qquad (3.23)$$

$$r(\mathbf{Q}, d) - P(\mathbf{Q}, E) = -Ed. \qquad (3.24)$$

We can now formulate two further functionals, the natural conditions of the first variations of which are expected to include the equations of motion in shallow water. The process is to use (3.23) to substitute for p in the integrand of (3.18) and (3.22) to substitute for r in the integrand of (3.19) by what is essentially a change of variables using (3.5). We note here that although (3.21) could be used to substitute for p in (3.18) and (3.24) could be used to substitute for r in (3.19) this would not change the nature of the functionals being generated. For instance integration by parts and the divergence theorem can be used on the 'P' functional generated by substituting (3.24) into (3.19) to give the functional formed by substituting (3.23) into (3.18).

The 'P' Principle

Let the functional $I_3(E, \mathbf{Q}, d, \phi)$ be defined by

$$\begin{aligned} I_3 = & \int_{t_1}^{t_2} \iint_D \left(P(\mathbf{Q}, E) - \mathbf{Q}.\boldsymbol{\nabla}\phi - d\,(E + \phi_t)\right)\, dx\, dy\, dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi \, d\Sigma\, dt + \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt \\ & + \iint_{D_\phi} \left(\left(d\,(\phi - h_2)\right)\big|_{t_2} - \left(d\,(\phi - h_1)\right)\big|_{t_1} \right) dx\, dy \\ & + \iint_{D_d} \left(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \right) dx\, dy. \end{aligned} \qquad (3.25)$$

The natural conditions of this 'P' principle

$$\delta I_3 = 0$$

are

$$\left.\begin{array}{rcl} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ P_E - d &=& 0 \\ E + \phi_t &=& 0 \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

$$\begin{array}{rcll} C - \mathbf{Q}.\mathbf{n} &=& 0 & \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\ \phi - f &=& 0 & \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\ d|_{t_i} - g_i &=& 0 & \text{in } D_d \text{ for } i = 1, 2, \\ \phi|_{t_i} - h_i &=& 0 & \text{in } D_\phi \text{ for } i = 1, 2. \end{array}$$

The first condition in Ω is

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0}.$$

Thus if equations (3.5) are assumed, the 'P' principle yields the conservation laws and the irrotationality condition as natural conditions in Ω and gives boundary conditions on ϕ and $\mathbf{Q}$ at space boundaries and on d and ϕ at time boundaries.

The 'R' Principle

Now consider a principle based on the function R. Let the functional $I_4(\mathbf{Q}, d, \mathbf{v}, \phi)$ be given by

$$\begin{aligned} I_4 = & \int_{t_1}^{t_2} \iint_D \left(-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + \phi\,(d_t + \boldsymbol{\nabla}.\mathbf{Q})\right)\,dx\,dy\,dt \\ & + \int_{t_1}^{t_2} \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\,d\Sigma\,dt - \int_{t_1}^{t_2} \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma\,dt \\ & - \iint_{D_d} \left((\phi\,(d - g_2))|_{t_2} - (\phi\,(d - g_1))|_{t_1}\right)\,dx\,dy \\ & - \iint_{D_\phi} \left(d|_{t_2}\,h_2 - d|_{t_1}\,h_1\right)\,dx\,dy. \end{aligned} \tag{3.26}$$

The natural conditions of this 'R' principle

$$\delta I_4 = 0$$

are

$$\left.\begin{array}{rcl} -R_{\mathbf{v}} + \mathbf{Q} &=& \mathbf{0} \\ -R_d - \phi_t &=& 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &=& \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &=& 0 \end{array}\right\} \quad \text{in } \Omega,$$

$$
\begin{aligned}
C - \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\
\phi - f &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\
d|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1, 2.
\end{aligned}
$$

The first two conditions in Ω may be written

$$
-d\mathbf{v} + \mathbf{Q} = \mathbf{0} \ , \ -gd - \frac{1}{2}\mathbf{v}.\mathbf{v} - \phi_t = 0.
$$

Thus the natural conditions of the 'R' principle include the equations of motion in shallow water and boundary conditions on the variables.

So there exists a quartet of functionals (3.18), (3.19), (3.25) and (3.26), based on the four functions p, r, P and R, from which the shallow water equations can be derived as the natural conditions of the first variations. Notice that the statements of the natural boundary conditions of all of the variational principles are identical and that the natural conditions in the domain are the same equations expressed in different variables.

3.6 Constrained and Reciprocal Principles

Variational principles can be constrained by assuming that the variations are made subject to the requirement that the variables satisfy one or more of the natural conditions. The principles constrained in this way will have the remaining natural conditions as natural conditions ([9]).

Reciprocal 'p' and 'r' Principles

The functional used in the 'p' principle (3.18) has an integrand which contains the integrated conservation of momentum equation and the irrotationality condition explicitly. It seems natural to constrain the 'p' principle to satisfy these two conditions. This can be done by specifying

$$
\left.
\begin{aligned}
E &= -\phi_t \\
\mathbf{v} &= \boldsymbol{\nabla}\phi
\end{aligned}
\right\}, \tag{3.27}
$$

which results in the functional I_1 reducing to a functional $\tilde{I}_1(\mathbf{Q}, d, \phi)$. The constrained principle is given by

$$
\begin{aligned}
\delta\tilde{I}_1 = \delta\Bigg\{ &\int_{t_1}^{t_2} \iint_D \hat{p}(\phi)\, dx\, dy\, dt + \int_{t_1}^{t_2} \int_{\Sigma_Q} C\phi\, d\Sigma\, dt \\
&+ \int_{t_1}^{t_2} \int_{\Sigma_\phi} (\phi - f)\, \mathbf{Q}.\mathbf{n}\, d\Sigma\, dt + \iint_{D_\phi} \Big((d\,(\phi - h_2))|_{t_2} - (d\,(\phi - h_1))|_{t_1} \Big)\, dx\, dy \\
&+ \iint_{D_d} \Big(\phi|_{t_2}\, g_2 - \phi|_{t_1}\, g_1 \Big)\, dx\, dy \Bigg\} = 0
\end{aligned} \tag{3.28}
$$

where

$$\hat{p}(\phi) = p(\boldsymbol{\nabla}\phi, -\phi_t) = \frac{1}{2g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)^2.$$

The natural conditions are

$$\left(-\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\right)_t + \boldsymbol{\nabla}.\left(-\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi\right) = 0 \qquad \text{in } \Omega,$$

$$\begin{aligned}
\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi.\mathbf{n} + C &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2],\\
\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi.\mathbf{n} + \mathbf{Q}.\mathbf{n} &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2],\\
\phi - f &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2],\\
\left.\left(\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right) + d\right)\right|_{t_i} &= 0 && \text{in } D_\phi \text{ for } i = 1, 2,\\
\phi|_{t_i} - h_i &= 0 && \text{in } D_\phi \text{ for } i = 1, 2,\\
\left.\frac{1}{g}\left(\phi_t + \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\right|_{t_i} + g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2,
\end{aligned}$$

the first of which may be recognised as conservation of mass, written in terms of ϕ, in the domain. Boundary conditions are given for ϕ, d and $\mathbf{Q}$.

If $\Sigma_Q = \Sigma$ and $D_d = D$ then (3.28) becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \hat{p}(\phi)\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma C\phi\,d\Sigma\,dt + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1\right)\,dx\,dy\right\} = 0 \qquad (3.29)$$

in which the functional depends on the variable ϕ alone.

A constrained 'r' principle can be constructed to satisfy conservation of mass by specifying

$$\left.\begin{aligned} d &= \boldsymbol{\nabla}.\boldsymbol{\psi}\\ \mathbf{Q} &= -\boldsymbol{\psi}_t \end{aligned}\right\}, \qquad (3.30)$$

for some vector $\boldsymbol{\psi} = \boldsymbol{\psi}(x, y, t)$, and substituting into equation (3.19). The resulting functional depends on $\boldsymbol{\psi}$ and ϕ and the variational principle is given by

$$\begin{aligned}
\delta\Bigg\{&\int_{t_1}^{t_2}\iint_D \hat{r}(\boldsymbol{\psi})\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_{\Sigma_Q} \phi\left(C + \boldsymbol{\psi}_t.\mathbf{n}\right)\,d\Sigma\,dt\\
&+ \int_{t_1}^{t_2}\int_{\Sigma_\phi} f\boldsymbol{\psi}_t.\mathbf{n}\,d\Sigma\,dt - \iint_{D_d}\left(\left(\phi\left(\boldsymbol{\nabla}.\boldsymbol{\psi} - g_2\right)\right)|_{t_2} - \left(\phi\left(\boldsymbol{\nabla}..\boldsymbol{\psi} - g_1\right)\right)|_{t_1}\right)\,dx\,dy\\
&- \iint_{D_\phi}\left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right)\,dx\,dy\Bigg\} = 0,
\end{aligned} \qquad (3.31)$$

where

$$\hat{r}(\boldsymbol{\psi}) = r(-\boldsymbol{\psi}_t, \boldsymbol{\nabla}.\boldsymbol{\psi}) = \frac{1}{2}\frac{\boldsymbol{\psi}_t.\boldsymbol{\psi}_t}{\boldsymbol{\nabla}.\boldsymbol{\psi}} - \frac{1}{2}g\left(\boldsymbol{\nabla}.\boldsymbol{\psi}\right)^2.$$

The natural conditions are

$$-\boldsymbol{\Psi}_t + \boldsymbol{\nabla}\left(g\boldsymbol{\nabla}.\boldsymbol{\psi} + \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi}\right) = \mathbf{0} \qquad \text{in } \Omega,$$

$$\begin{aligned}
C + \boldsymbol{\psi}_t.\mathbf{n} &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\
-\phi_t - \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi} - g\boldsymbol{\nabla}.\boldsymbol{\psi} &= 0 && \text{on } \Sigma_Q \text{ for } t \in [t_1, t_2], \\
-f_t - \frac{1}{2}\boldsymbol{\Psi}.\boldsymbol{\Psi} - g\boldsymbol{\nabla}.\boldsymbol{\psi} &= 0 && \text{on } \Sigma_\phi \text{ for } t \in [t_1, t_2], \\
\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_i} - g_i &= 0 && \text{in } D_d \text{ for } i = 1, 2, \\
\boldsymbol{\nabla}\phi|_{t_i} + \boldsymbol{\Psi}|_{t_i} &= \mathbf{0} && \text{in } D_d \text{ for } i = 1, 2, \\
\boldsymbol{\nabla} h_i + \boldsymbol{\Psi}|_{t_i} &= \mathbf{0} && \text{in } D_\phi \text{ for } i = 1, 2, \\
\phi|_{t_i} - h_i &= 0 && \text{on } \Sigma_Q \cap \Sigma_{\hat{\phi}} \text{ for } i = 1, 2, \\
\phi|_{t_i} - \phi|_{t_i} &= 0 && \text{on } \Sigma_Q \cap \Sigma_d \text{ for } i = 1, 2, \\
f_i - h_i &= 0 && \text{on } \Sigma_\phi \cap \Sigma_{\hat{\phi}} \text{ for } i = 1, 2, \\
f_i - \phi|_{t_i} &= 0 && \text{on } \Sigma_\phi \cap \Sigma_d \text{ for } i = 1, 2,
\end{aligned}$$

where $\Sigma_{\hat{\phi}}$ is the boundary of D_ϕ and Σ_d is the boundary of D_d and, for neatness, $\boldsymbol{\Psi}$ represents the term $\boldsymbol{\psi}_t/\boldsymbol{\nabla}.\boldsymbol{\psi}$.

The natural condition in D is recognisable as the equation of conservation of momentum — not the integrated form usually generated. The irrotationality condition can be derived as a consequence of the conservation of momentum and the boundary conditions which specify that the flow is irrotational at $t = t_1$.

If $\Sigma_\phi = \Sigma$ and $D_\phi = D$, (3.31) reduces to a variational principle involving a functional of $\boldsymbol{\psi}$ alone, namely,

$$\delta\left\{\int_{t_1}^{t_2}\iint_D \hat{r}(\boldsymbol{\psi})\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma f\boldsymbol{\psi}_t.\mathbf{n}\,d\Sigma\,dt - \iint_D \left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right)dx\,dy\right\} = 0. \tag{3.32}$$

The variational principles (3.29) and (3.32) can be described as reciprocal. We use this term to mean that the constraints satisfied in the domain by the variations in one principle are the natural conditions, in the domain, of the other principle. The boundary conditions also exhibit reciprocity in that the natural boundary conditions of (3.29) are given for mass flow, as a function of ϕ, on Σ and depth, as a function of ϕ, in D for $t = t_1$, t_2 whereas in (3.32) conditions are for the energy, as a function of $\boldsymbol{\psi}$, on Σ and velocity, as a function of $\boldsymbol{\psi}$, in D for $t = t_1$, t_2.

Reciprocal 'P' and 'R' Principles

Now consider the other two variational principles — based on P and R. The integrands of the 'P' and 'R' functionals are not expressible in the form

$$P \text{ or } R \text{ function} + \text{multiplier} \times \text{conservation law}$$

so there is no corresponding way of constraining the variational principles and the functionals cannot in the same way be reduced to depend on one variable. However, the following structure can be deduced.

Consider the 'P' functional (3.25). Let $\Sigma_Q = \Sigma$ and $D_d = D$, and constrain the variables to satisfy conservation of momentum using the first of (3.27). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D (P(\mathbf{Q}, -\phi_t) - \mathbf{Q}.\boldsymbol{\nabla}\phi)\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma C\phi\,d\Sigma\,dt \right.$$
$$\left. + \iint_D \left(\phi|_{t_2} g_2 - \phi|_{t_1} g_1\right) dx\,dy\right\} = 0, \qquad (3.33)$$

where the variables are $\mathbf{Q}$ and ϕ. The natural conditions are given by

$$\left.\begin{aligned} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ (P_{-\phi_t})_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } \Omega,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma \text{ for } t \in [t_1, t_2], \\ g_i - P_{-\phi_t}|_{t_i} &= 0 \qquad \text{in } D \text{ for } i = 1, 2. \end{aligned}$$

The first two conditions may be rewritten as

$$\left.\begin{aligned} \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ d_t + \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } \Omega,$$

which are the irrotationality condition and the conservation of mass equation.

In the 'R' functional (3.26) let $\Sigma_\phi = \Sigma$ and $D_\phi = D$ and constrain the variations to satisfy conservation of mass by imposing (3.30). Then the variational principle becomes

$$\delta\left\{\int_{t_1}^{t_2}\iint_D (-R(\mathbf{v}, \boldsymbol{\nabla}.\boldsymbol{\psi}) - \boldsymbol{\psi}_t.\mathbf{v})\,dx\,dy\,dt + \int_{t_1}^{t_2}\int_\Sigma f\boldsymbol{\psi}_t.\mathbf{n}\,d\Sigma\,dt \right.$$
$$\left. - \iint_D \left(\boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_2} h_2 - \boldsymbol{\nabla}.\boldsymbol{\psi}|_{t_1} h_1\right) dx\,dy\right\} = 0, \qquad (3.34)$$

which involves a functional of $\mathbf{v}$ and $\boldsymbol{\psi}$. The natural conditions are given by

$$\left.\begin{aligned} -R_{\mathbf{v}} - \boldsymbol{\psi}_t &= \mathbf{0} \\ \boldsymbol{\nabla}R_{\boldsymbol{\nabla}.\boldsymbol{\psi}} + \mathbf{v}_t &= \mathbf{0} \end{aligned}\right\} \qquad \text{in } \Omega,$$

$$
\begin{aligned}
-R_{\boldsymbol{\nabla}.\boldsymbol{\psi}} - f_t &= 0 && \text{on } \Sigma \text{ for } t \in [t_1, t_2], \\
\boldsymbol{\nabla} h_i - \mathbf{v}|_{t_i} &= \mathbf{0} && \text{in } D \text{ for } i = 1, 2, \\
f|_{t_i} - h_i &= 0 && \text{on } \Sigma \text{ for } i = 1, 2.
\end{aligned}
$$

The first two equations are

$$
\left.
\begin{aligned}
-(\boldsymbol{\nabla}.\boldsymbol{\psi})\,\mathbf{v} - \boldsymbol{\psi}_t &= \mathbf{0} \\
\mathbf{v}_t + \boldsymbol{\nabla}\left(g\boldsymbol{\nabla}.\boldsymbol{\psi} + \tfrac{1}{2}\mathbf{v}.\mathbf{v}\right) &= \mathbf{0}
\end{aligned}
\right\} \qquad \text{in } \Omega,
$$

the second of which is conservation of momentum. This, together with the natural condition in D for t_1, implies the irrotationality condition in D for $t \in [t_1, t_2]$.

The constrained 'P' and 'R' principles (3.33) and (3.34) are reciprocal since the constraint of conservation of momentum in (3.33) is a natural condition of (3.34) and the conservation of mass constraint in (3.34) is a natural condition of (3.33). The irrotationality condition is a natural condition of both principles.

4 Variational Principles for Steady Shallow Water Flows

The discussion so far has concerned derivation of variational principles whose natural conditions include the unsteady shallow water equations of motion developed both from principles whose natural conditions are the equations of motion of a free surface flow and independently. We now seek to apply these principles to steady state conditions.

The steady state equations of motion in shallow water for a domain with constant undisturbed depth can be deduced from the unsteady equations (3.4). The steady state condition assumes that all of the flow variables — mass flow, energy, depth and velocity — are independent of time. The potential ϕ is not a flow variable and cannot therefore be assumed to be independent of time although its general form can be deduced.

The irrotationality condition is

$$
\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0}.
$$

Using this together with $(3.5)_2$ we may write the integrated version of the conservation of momentum equation as

$$
E + \phi_t = 0.
$$

Assuming steady state conditions we have

$$
E_t = 0 \ , \ \mathbf{v}_t = \mathbf{0}
$$

and thus

$$\phi_{tt} = 0 \ , \ \boldsymbol{\nabla}\phi_t = \mathbf{0}.$$

Therefore ϕ is of the form

$$\phi(x, y, t) = -\hat{E}t + \tilde{\phi}(x, y) \tag{4.1}$$

where the energy $\hat{E}$, the steady counterpart of E, is a constant. The expression (4.1) will prove useful in reducing functionals for unsteady motion to functionals for steady motion.

The steady state equations are given by

$$\begin{aligned} \boldsymbol{\nabla}.\mathbf{Q} &= 0 && \text{conservation of mass} && (4.2)\\ \boldsymbol{\nabla}\hat{E} &= \mathbf{0} && \text{conservation of momentum} && (4.3)\\ \mathbf{v} &= \boldsymbol{\nabla}\phi && \text{irrotationality,} && (4.4) \end{aligned}$$

where $\mathbf{Q}$ and $\hat{E}$ are given by

$$\begin{aligned} \mathbf{Q} &= d\mathbf{v} && (4.5)\\ \hat{E} &= gd + \frac{1}{2}\mathbf{v}.\mathbf{v}. && (4.6) \end{aligned}$$

The equation of conservation of momentum is satisfied exactly since the energy $\hat{E}$ is a constant.

4.1 Steady Principles from Unsteady Principles

Consider the four functionals (3.18), (3.19), (3.25) and (3.26). The boundary functions $C(x, y, t)$, $f(x, y, t)$, $g_i(x, y)$ and $h_i(x, y)$ $(i = 1, 2)$ must be treated with care in transforming from unsteady to steady motion. In the natural conditions the function C will be used to provide a boundary condition for the mass flow on Σ_Q. As mass flow is now assumed independent of time, C must be replaced by a function $\hat{C}(x, y)$. The function f will be used to provide a boundary condition for ϕ on Σ_ϕ. The variation of ϕ with time is known from (4.1) and so, for consistency, f must be replaced by $\tilde{f}(x, y, t) = -\hat{E}t + \tilde{f}_1(x, y)$. The functions g_1 and g_2 are the time boundary conditions on the depth in domain D_d but since the depth does not vary with time we must have $g_1 = g_2 = \hat{g}(x, y)$. The functions h_1 and h_2 give time boundary conditions on ϕ in D_ϕ, and from (4.1) we know that

$$\phi\big|_{t_2} - \phi\big|_{t_1} = -\hat{E}(t_2 - t_1) = -\hat{E}T,$$

where $T = t_2 - t_1$. Therefore h_1 and h_2 must be specified so that $h_2 - h_1 = -\hat{E}T$ for consistency.

First consider the 'p' principle. The functional I_1, given by (3.18), under steady state conditions becomes

$$I_1^s = \iint_D T\left(p(\mathbf{v},\hat{E}) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\tilde{\phi}\right)\right) dx\,dy + \int_{\Sigma_Q} \hat{C}\left(\int_{t_1}^{t_2} \phi\,dt\right) d\Sigma$$
$$+ \int_{\Sigma_\phi} \left(\int_{t_1}^{t_2} \left(\phi - \tilde{f}\right) dt\right) \mathbf{Q}.\mathbf{n}\,d\Sigma - \iint_{D_d} T\hat{g}\hat{E}\,dx\,dy,$$

where $I_1^s = I_1^s(\mathbf{Q},\mathbf{v},\phi)$. To simplify this define

$$\hat{f}(x,y) = \tilde{f}_1(x,y) - \frac{1}{2}\hat{E}(t_1+t_2)$$

so that

$$\int_{t_1}^{t_2} \tilde{f}(x,y,t)\,dt = \left[\tilde{f}_1(x,y)t - \frac{1}{2}\hat{E}t^2\right]_{t_1}^{t_2} = T\left(\tilde{f}_1(x,y) - \frac{1}{2}\hat{E}(t_1+t_2)\right) = T\hat{f}(x,y).$$

Also, let

$$\hat{\phi}(x,y) = \tilde{\phi}(x,y) - \frac{1}{2}\hat{E}(t_1+t_2)$$

so that

$$\int_{t_1}^{t_2} \phi(x,y,t)\,dt = \left[\tilde{\phi}(x,y)t - \frac{1}{2}\hat{E}t^2\right]_{t_1}^{t_2} = T\left(\tilde{\phi}(x,y) - \frac{1}{2}\hat{E}(t_1+t_2)\right) = T\hat{\phi}(x,y)$$

and $\boldsymbol{\nabla}\hat{\phi} = \boldsymbol{\nabla}\tilde{\phi}$. Then

$$I_1^s = \iint_D T\left(p(\mathbf{v},\hat{E}) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\hat{\phi}\right)\right) dx\,dy + \int_{\Sigma_Q} T\hat{C}\hat{\phi}\,d\Sigma$$
$$+ \int_{\Sigma_\phi} T\left(\hat{\phi} - \hat{f}\right) \mathbf{Q}.\mathbf{n}\,d\Sigma - \iint_{D_d} T\hat{g}\hat{E}\,dx\,dy. \qquad (4.7)$$

Notice that the final term in (4.7) is a constant and so it may be discarded. Also, throughout the functional there is a constant non-zero multiplier T which may be set equal to unity. Finally, for neatness, the ˆ notation is suppressed and the 'p' functional for use in the steady state variational principle is written as

$$L_1 = \iint_D \left(p(\mathbf{v},E) + \mathbf{Q}.\left(\mathbf{v} - \boldsymbol{\nabla}\phi\right)\right) dx\,dy + \int_{\Sigma_Q} C\phi\,d\Sigma + \int_{\Sigma_\phi} (\phi - f)\,\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.8)$$

where $L_1 = L_1(\mathbf{Q},\mathbf{v},\phi)$.

By the same process steady state forms of (3.19), (3.25) and (3.26) can be generated, and using the method by which (4.8) was deduced from (4.7) the steady state 'r', 'P' and 'R' functionals may be written

$$L_2 = \iint_D \left(r(\mathbf{Q},d) + Ed + \phi\boldsymbol{\nabla}.\mathbf{Q}\right) dx\,dy$$
$$+ \int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\,d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.9)$$

$$L_3 = \iint_D \left(P(\mathbf{Q},E) + \phi\boldsymbol{\nabla}.\mathbf{Q}\right) dx\,dy$$

$$+\int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\,d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.10)$$

$$L_4 = \iint_D (-R(\mathbf{v}, d) + \mathbf{Q}.\mathbf{v} + Ed + \phi\boldsymbol{\nabla}.\mathbf{Q})\,dx\,dy$$

$$+\int_{\Sigma_Q} \phi\,(C - \mathbf{Q}.\mathbf{n})\,d\Sigma - \int_{\Sigma_\phi} f\mathbf{Q}.\mathbf{n}\,d\Sigma, \qquad (4.11)$$

where $L_2 = L_2(\mathbf{Q}, d, \phi)$, $L_3 = L_3(\mathbf{Q}, \phi)$ and $L_4 = L_4(\mathbf{Q}, d, \mathbf{v}, \phi)$.

The natural conditions of the steady state variational principles

$$\delta L_1 = \delta L_2 = \delta L_3 = \delta L_4 = 0$$

are expected to include the shallow water equations of motion (4.2) and (4.4) and possibly (4.5) or (4.6). Equation (4.3) is satisfied exactly since the energy E is regarded as a given constant.

The natural conditions of $\delta L_1 = 0$, the 'p' principle, are

$$\left.\begin{aligned} p_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_Q, \\ \phi - f &= 0 \quad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$-\frac{\mathbf{v}}{\mathbf{g}}\left(E - \frac{1}{2}\mathbf{v}.\mathbf{v}\right) + \mathbf{Q} = \mathbf{0} \qquad \text{in } D.$$

The natural conditions of $\delta L_2 = 0$, the 'r' principle, are

$$\left.\begin{aligned} r_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ r_d + E &= 0 \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \quad \text{on } \Sigma_Q, \\ \phi - f &= 0 \quad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} \frac{\mathbf{Q}}{\mathbf{d}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ -\tfrac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d} - gd + E &= 0 \end{aligned}\right\} \quad \text{in } D.$$

The natural conditions of $\delta L_3 = 0$, the 'P' principle, are

$$\left.\begin{aligned} P_{\mathbf{Q}} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \quad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ \phi - f &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first equation being

$$\mathbf{v} - \boldsymbol{\nabla}\phi = \mathbf{0} \qquad \text{in } D,$$

where $\mathbf{v}$ is a function of $\mathbf{Q}$ and E using (4.5) and (4.6).

The natural conditions of $\delta L_4 = 0$, the 'R' principle, are

$$\left.\begin{aligned} -R_{\mathbf{v}} + \mathbf{Q} &= \mathbf{0} \\ -R_d + E &= 0 \\ \mathbf{v} - \boldsymbol{\nabla}\phi &= \mathbf{0} \\ \boldsymbol{\nabla}.\mathbf{Q} &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

$$\begin{aligned} C - \mathbf{Q}.\mathbf{n} &= 0 \qquad \text{on } \Sigma_Q, \\ \phi - f &= 0 \qquad \text{on } \Sigma_\phi, \end{aligned}$$

the first two equations being

$$\left.\begin{aligned} -d\mathbf{v} + \mathbf{Q} &= \mathbf{0} \\ -gd - \tfrac{1}{2}\mathbf{v}.\mathbf{v} + E &= 0 \end{aligned}\right\} \qquad \text{in } D.$$

Thus the natural conditions of the steady state motion variational principles derived from free surface unsteady motion variational principles include the steady state equations in shallow water — (4.2) and (4.4). In order that the equations are expressed in the form of (4.2) and (4.4) it is necessary to assume in the 'p' principle that $d = (E - 1/2\mathbf{v}.\mathbf{v})/g$, in the 'r' principle that $\mathbf{v} = \mathbf{Q}/d$ and in the 'P' principle that $E = gd + 1/2\mathbf{v}.\mathbf{v}$ and $\mathbf{Q} = d\mathbf{v}$.

Incidentally the same 'p' and 'r' functionals (4.8) and (4.9) can be derived from the 'pressure' and 'Hamilton' free surface functionals by a different route. Instead of applying the shallow water approximation and then considering steady flow the assumption of steady state conditions can be made first. The method is dependent on the addition of appropriate boundary terms to the free surface functionals for unsteady flow in the same way that boundary terms were added to the functionals for unsteady flow in shallow water in Subsection 3.4.

4.2 Constrained and Reciprocal Principles

The variational principles for steady motion can be constrained in the same way as the ones for unsteady motion were in Section 3. For the unsteady variational principles there were three natural conditions which could be used as constraints — singly or in pairs — conservation of mass, conservation of momentum and the irrotationality condition. For the steady variational principles there are just two — conservation of mass and the irrotationality condition — since conservation of momentum is satisfied implicitly.

Reciprocal 'p' and 'P' Principles

Consider the integrands of the functionals (4.8)–(4.11). In Section 3 emphasis was placed on the structure of the integrands of the 'p' and 'r' functionals — they were expressed as a function plus a multiple of a conservation law or the irrotationality condition. For steady flows the 'p' and 'P' functionals also exhibit this property, so that the 'p' principle will be constrained by irrotationality and the 'P' principle by conservation of mass.

Let $\Sigma_\phi = \Sigma$ and $\phi = f$ on Σ. Then the 'p' principle constrained by irrotationality is a functional of ϕ alone and is given by

$$\delta \left\{ \iint_D p(\boldsymbol{\nabla}\phi, E)\, dx\, dy \right\} = 0, \tag{4.12}$$

where ϕ is constrained by $\phi = f$ on Σ, with the natural condition

$$\boldsymbol{\nabla}.\left(\frac{1}{g}\left(E - \frac{1}{2}\boldsymbol{\nabla}\phi.\boldsymbol{\nabla}\phi\right)\boldsymbol{\nabla}\phi\right) = 0 \qquad \text{in } D$$

which is conservation of mass.

Let $\Sigma_Q = \Sigma$ and $\mathbf{Q}.\mathbf{n} = C$ on Σ. Then the constrained 'P' principle is a functional of $\mathbf{Q}$ alone and is given by

$$\delta \left\{ \iint_D P(\mathbf{Q}, E)\, dx\, dy \right\} = 0, \tag{4.13}$$

where $\mathbf{Q}$ is constrained by $\boldsymbol{\nabla}.\mathbf{Q} = 0$ and by $\mathbf{Q}.\mathbf{n} = C$ on Σ, which has as its only natural condition the irrotationality condition.

Thus the 'p' and 'P' steady principles display the same relationship as the 'p' and 'r' principles for unsteady flow (3.29) and (3.32) — they are both functionals of one variable and are reciprocal in the sense defined earlier.

The particular relationship between the 'p' and 'r' principles (3.29) and (3.32) for unsteady flow has not survived the transition to principles for steady flow. The 'p' and 'r' functionals (4.8) and (4.9) cannot be constrained so that they each depend on just one function and have reciprocal constraints and natural conditions. The relationship of the 'p' and 'r' principles in unsteady motion is a result of the integrands being expressible in the form

$$p \text{ or } r \text{ function} + \text{multiplier} \times \text{conservation law}$$

and that once the variables are constrained to satisfy the relevant conservation law and, in the case of the p function, irrotationality the functions p and r can each be written in terms of one variable. In the steady motion functional (4.8) the pressure function p is still expressed as a function of $\mathbf{v}$ and E but, since E is a constant, p is in fact a function of $\mathbf{v}$ alone. The flow stress P is also a function of one variable so that the constrained 'p' and 'P' principles for steady motion exhibit the same relationship as the constrained 'p' and 'r' principles for unsteady motion in terms of being reciprocal and depending on just one variable.

Reciprocal 'r' and 'R' Principles

The function r depends on $\mathbf{Q}$ and d and cannot be written as a function of one variable by requiring the irrotationality condition or conservation of mass to be satisfied. The function R also depends on two variables and cannot be reduced to a function of one variable. However the 'r' and 'R' principles for steady motion can be constrained to give reciprocal principles.

Let $\Sigma_\phi = \Sigma$ and $\phi = f$ on Σ. Then constraining the 'R' principle to satisfy the irrotationality condition gives

$$\delta\left\{\iint_D (-R(\boldsymbol{\nabla}\phi, d) + Ed)\, dx\, dy\right\} = 0 \tag{4.14}$$

which depends on ϕ and d. The variational principle (4.14) has natural conditions

$$\left.\begin{aligned} -R_d + E &= 0 \\ -\boldsymbol{\nabla}.\left(R_{\boldsymbol{\nabla}\phi}\right) &= 0 \end{aligned}\right\} \qquad \text{in } D,$$

the second of which is conservation of mass in the form

$$\boldsymbol{\nabla}.(d\boldsymbol{\nabla}\phi) = 0.$$

Let $\Sigma_Q = \Sigma$ and $\mathbf{Q}.\mathbf{n} = C$ on Σ then constraining the 'r' principle to satisfy conservation of mass gives

$$\delta\left\{\iint_D (r(\mathbf{Q}, d) + Ed)\, dx\, dy\right\} = 0, \tag{4.15}$$

where $\boldsymbol{\nabla}.\mathbf{Q} = 0$, which has as natural conditions

$$r_d + E = 0$$

and the irrotationality condition in D.

The 'r' and 'R' principles are reciprocal since the constraint of one principle is a natural condition of the other. Unlike the constrained 'p' and 'P' principles though, the 'r' and 'R' principles are functionals of two variables which yield a second natural condition for each.

The 'p' and 'r' principles, (4.12) and (4.15), and the 'P' and 'R' principles, (4.13) and (4.14), may still be thought of as reciprocal pairs, in the sense defined earlier, as for the corresponding constrained principles for unsteady motion.

5 Concluding Remarks

In this final section, some further connections between existing literature and the contents of this report are mentioned and a few remarks about practical implementation of the variational principles are made.

In Section 3 four functionals, related to one another by Legendre transformations, are defined. These are the pressure function p, the Lagrangian density r, the flow stress P and the function R which, when integrated over the space domain, gives the total energy of the flow. In [10] Benjamin and Bowman consider conservation laws and symmetry properties of Hamiltonian systems including shallow water, for which they derive four functions. Two of these — identified by them as a Hamiltonian density and a flow force — have the values of the functions R and P respectively, apart from constant multipliers.

That the function R is indeed the Hamiltonian density for shallow water flow can be seen by considering the 'Legendre transformation' of the Lagrangian L. Let

$$L(\mathbf{Q}, d) = \iint_D r(\mathbf{Q}, d)\, dx\, dy \tag{5.16}$$

where $r(\mathbf{Q}, d)$ is the Lagrangian density. Then the functional derivatives $\partial L/\partial d$ and $\partial L/\partial \mathbf{Q}$ can be defined by

$$\delta L = \left(\delta d, \frac{\partial L}{\partial d}\right) + \left(\delta \mathbf{Q}, \frac{\partial L}{\partial \mathbf{Q}}\right),$$

where the inner product (α, β) is given by

$$(\alpha, \beta) = \iint_D \alpha\beta\, dx\, dy.$$

Thus

$$\frac{\partial L}{\partial d} = -\frac{1}{2}\frac{\mathbf{Q}.\mathbf{Q}}{d^2} - gd \;, \quad \frac{\partial L}{\partial \mathbf{Q}} = \frac{\mathbf{Q}}{d} = \mathbf{v}.$$

The Hamiltonian $H(\mathbf{v}, d)$ is given, in a form somewhat analogous to (1.2), by

$$H(\mathbf{v}, d) = (\mathbf{v}, \mathbf{Q}) - L(\mathbf{Q}, d),$$

and thus it is easily seen that

$$H(\mathbf{v}, d) = \iint_D R(\mathbf{v}, d)\, dx\, dy$$

where R is the Hamiltonian density.

The analogy with (1.2) would have been closer if we had started with a Lagrangian $L(\mathbf{v}, d)$, with density $r(d\mathbf{v}, d)$, and deduced a Hamiltonian $H(\mathbf{Q}, d)$ with density $R(\mathbf{Q}/d, d)$. This alteration identifies $\mathbf{v}$ as the vector of generalised velocities and $\mathbf{Q} = d\mathbf{v}$ as the vector of conjugate momenta. We can, of course, redefine the Lagrangian and Hamiltonian densities in terms of the variables $\mathbf{v}$, d and $\mathbf{Q}$, d respectively, to achieve the structure mentioned. The redefined densities are themselves linked by a Legendre transformation with $\mathbf{v}$ and $\mathbf{Q}$ active and d passive, but the quartet of densities obtained by taking $r(d\mathbf{v}, d)$ as the starting point does not include a quantity identifiable as the flow stress or the energy E as an independent variable. In particular, the contact with the Bateman functions is then lost. Moreover, Benjamin and Bowman's [10] Hamiltonian density for shallow flow is expressed in terms of the speed v, rather than Q.

Although it is by no means a complete catalogue of all of the possibilities, a selection of constrained variational principles is described in Subsections 3.6 and 4.2. The 'p' principle for steady flow constrained to satisfy irrotationality and a boundary condition on the velocity potential (equation (4.12)) and its reciprocal principle (4.13) — the 'P' principle constrained to satisfy conservation of mass and a boundary condition on the mass flow — are examples of Bateman's functions [7], using the gas dynamics analogy. Sewell [11] re-examined the relationships between these principles, in the context of Legendre transformations, for three-dimensional steady flows in perfect fluids.

In the variational principles for unsteady shallow water flow an element of choice is introduced by dividing the domain into the two distinct regions D_d and D_ϕ and the boundary into two distinct sections Σ_ϕ and Σ_Q, as defined in Subsection 3.4. For practical purposes it is possible to define D_d, D_ϕ, Σ_ϕ and Σ_Q to suit the available boundary conditions, which must be for depth and/or velocity potential in the domain for $t = t_1, t_2$ and for velocity potential and/or mass flow on the boundary for $t \in [t_1, t_2]$.

The variational principles for steady shallow water flow require conditions, for mass flow and/or the velocity potential, on the boundary of the domain only. In this case, boundary conditions for depth and velocity can also be accommodated by using (4.5) and (4.6) to give a corresponding boundary condition for mass flow, since the constant energy E is known.

For both unsteady and steady shallow water flows constrained variational principles which depend on only one function are available. This suggests that there is practical potential for using the constrained variational principles to solve for such flows approximately by numerical methods, and this approach will be taken in a later report.

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